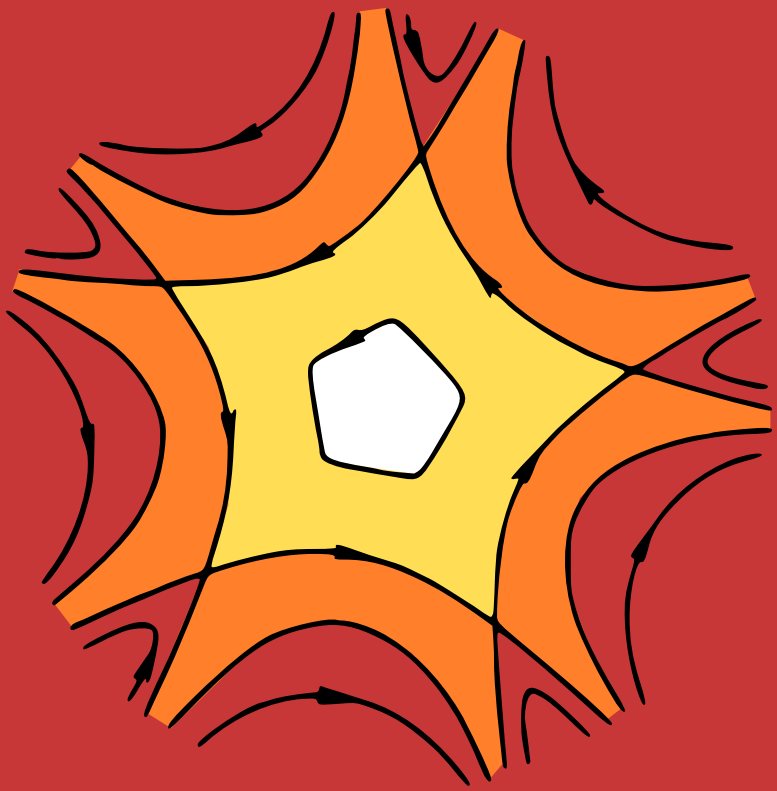
A.A. Kolemsky

Theory of Cyclic Accelerators



THEORY OF CYCLIC ACCELERATORS

THEORY OF CYCLIC ACCELERATORS

A. A. KOLOMENSKY

AND

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M. BARBIER (CERN)

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PREFACE

The impetuous advance of the physics and techniques of accelerators stands out by its rapidity from the background of the general rapid progress of nuclear and elementary particle physics. The acceleration of particles to high energies has given rise to a new specialised branch of science in which the most varied questions of theoretical and experimental physics and techniques are tightly woven together. The results in this field are obvious: for example, it took only thirty years for cyclical accelerators to travel the whole way from the first table cyclotron to the giant machines with which particles are brought to some tens of milliards* of electron volt, energies which sounded fantastic not so long ago. Large accelerators have been constructed in various countries, and laboratories and institutes occupying large communities of physicists and engineers have been built around them.

All these achievements have been reflected in an extremely large quantity of papers of theoretical and experimental character which have appeared in the periodical literature. To find one's way through this enormous material is in itself no easy task. For this reason a series of books and reviews has given some help in their time, describing the physical principles and basic data pertinent to the construction of accelerators. However, the majority of these accounts are getting old nowadays and, what is most important, are of a descriptive nature and do not give in a logical manner the basis of the theory, which has undergone much development in recent years and has become more complicated as new types of accelerators appeared. For this reason, without doubt, the need has ripened for books in which the many theoretical investigations on accelerators are generalised and systematised.

In writing this book the authors hoped to fill this gap to some extent and give a more or less systematic presentation of the theory of motion

* Milliard = 10^9 (Billion in U.S.A.).

of particles in present-day accelerators from a unified point of view. The main effort in the book is put into the theory of the new types of cyclic accelerators, primarily those using strong focusing. Less space is given to the 'old' accelerators, such as cyclotrons and betatrons, as they are rather well described in plenty of textbooks and monographs and, furthermore, because a more detailed presentation of their theory would have led to a considerable increase in the size of the book.

The book is intended for scientists and engineers working in high energy particle physics and on related topics, and also for students and graduates in the corresponding specialized fields. As we consider that the various questions developed may be of interest to others than those specially occupied with accelerators and their applications, we have also given briefly in the book basic information on the construction of accelerators, which will help understanding of the meaning of the presented theory. For this reason we have considered it appropriate to provide a broad introductory chapter in which, besides an account of the historical development of accelerators and their role in contemporary physics, we give an elementary introduction to the theory, which will be developed in the following chapters.

Chapters 2–4 constitute the basic material of the book and contain a systematic presentation of the general theory of particle motion in cyclic accelerators. Special attention is paid to methods of study of the various perturbations encountered in practice. Besides questions having immediate application to existing types of accelerator, particular care is taken to treat those presenting definite interest from the point of view of principles.

The final chapter 7 is devoted to peculiarities of accelerators of various types. First, we present some important details and practical approximations, which it was convenient to leave out of the chapter containing general theoretical considerations. Secondly, we study in this chapter those types of accelerator whose principles it would have been difficult to present with the general theory without disturbing the flow of the treatment. Here belong systems which have just begun to be used in recent times (strong focusing fixed field accelerators, storage systems, stochastic methods, etc.).

The contents of the book basically cover in full the general theory of cyclic accelerators at a given stage of their development. We have not,

however, touched certain problems which, although important, are still far from their complete solution and demand further development and comparison with experiment. Here belong automatic regulation by the accelerator itself using the beam of accelerated particles, influence of space charge, etc.

Furthermore, we did not give the theory of injection and ejection of particles into and out of accelerators, as the solution of these questions depends a great deal on the type and scale of the apparatus and on other practical aspects. Their presentation would have increased the volume of the book considerably.

Now a few words about the character of the presentation. In all chapters we have tried to explain the various phenomena as well as the characteristics of the various types of accelerator on the basis of a unified theory. In only a few cases have we considered it appropriate to give possible complementary variants of the theory, primarily where this permitted a greater completeness in obtaining results and made it possible to look at a given question from a more general point of view. For instance, the use of the canonical form of the equation of motion will serve this purpose when investigating betatron oscillations (ch. 2 § 5) and acceleration regimes (ch. 7 § 5).

References to the literature are given in the book, as usual, covering both papers necessary for the basic development of a subject, and papers proposing new acceleration methods. This, of course, does not mean that other papers do not contain valuable and interesting contributions. Often, when presenting a question, it has not been possible to be in accord with a publication or a series of publications, so that it was necessary to reconsider the matter with a view to conserving the systematics of presentation. Part of the book is based on the original papers by the authors.

Although the book is intended for a relatively wide circle of readers, the level of difficulty of its different parts may be quite varied. For instance, the paragraphs printed in small type, which contain more precise or supplementary statements added to the basic text, could be left aside at first reading. In the appendices some longer complements and proofs of general theorems are given.

The authors would like to express their heartfelt thanks to academician V. I. Veksler, Prof. M. S. Rabinovitch, I.S. Danylkyn, V. I. Kotov and

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LIST OF SYMBOLS

In each chapter the formulae are numbered independently within each section (1.1), (1.2),... (2.1), (2.2),... and so on. The first figure quoted indicates the number of the section, the second one the number of the formula within this section. Reference to formulae in a given section is made without quoting the section number. Reference to formulae in another chapter is made giving the chapter number, and to formulae belonging to appendices, giving their title, e.g. (A.4).

In the following are listed the most important symbols which, as one would expect, are the same throughout all chapters.

A – amplitude of oscillation

$\boldsymbol{B}, B_x, B_y, B_z$ – magnetic induction vector and its components

$\mathscr{D}$ – Wronskian of the solution

E, E_0 – total energy and rest energy of the particle

E_k – critical energy

$\boldsymbol{\mathscr{E}}$ – electrical field strength

F – form factor of the oscillation

$\mathscr{H}$ – Hamiltonian

I – variable part of the integral of action

K – curvature

$\mathscr{K}$ – coefficient determined by the self-focusing properties of the accelerator $\Delta T/T = \mathscr{K}(\Delta E/E)$

$\mathbf{M}$ – transfer matrix in a periodic system

N – number of periods

$\mathbf{P}$ – matrix of the system of linear equations

P – generalized momentum

Q – generalized co-ordinate

R – radius of the orbit in an azimuthally symmetrical field

T – period of particle revolution

T_{acc} – acceleration time

$\boldsymbol{V}$ – eigenvector of matrix **M**
V – amplitude of accelerating voltage
W – kinetic energy or power of radiative loss (ch. 5)
$\boldsymbol{X}$ – vector of the oscillation
a – complex amplitude
$\boldsymbol{b}$ – unit vector along the binormal (ch. 2) or perturbation of the magnetic field
c – velocity of light
e – particle charge
$f(\vartheta)$ – Floquet's function
$g(\vartheta)$ – coefficient in Hill's equation (focusing function)
$\hbar$ – Planck's constant divided by 2π
l – length of straight section
m, m_0 – total mass and rest mass of the particle
$\boldsymbol{n}$ – unit vector along the normal
n – index of the magnetic field
p – momentum of the particle
$q = \omega_0/\omega_s$ – harmonic order
$\boldsymbol{r}$ – radius vector
$r_e = e^2/m_0 c^2$ – classical radius of the electron
s – index indicating quantities at equilibrium or related to synchrotron oscillations (for instance ω_s)
t – time
v – particle velocity
x – displacement along the normal to the orbit
x_0 – closed-orbit displacement
z – displacement from the orbit along the binormal (vertical)
α – orbit expansion coefficient ($(\Delta\Pi/\Pi) = \alpha(\Delta p/p)$)
$\beta = v/c$ – ratio of particle velocity to the velocity of light
Γ – dimensionless power of radiative loss ($\Gamma = W/E\omega$), ch. 5
$\gamma = (1-\beta^2)^{-\frac{1}{2}}$ – relativistic factor
$\varepsilon = \hbar\omega$ – photon energy (ch. 5)
ζ – decrement of oscillation (ch. 5)
$\eta = \varphi - \varphi_s$ – shift from the stable-phase angle
θ – azimuthal angle in cylindrical co-ordinates
ϑ – generalized azimuth

χ – torsion

Λ – eigenvalue of Hill's equation, $\Lambda = \hbar/m_0 c$ – Compton wave length

λ – eigenvalue of transfer matrix **M** or relative length of the straight sections (ch. 7).

μ – characteristic figure

ν_x, ν_z, ν_s – betatron and synchrotron oscillation frequencies (with respect to a generalized azimuth)

Π – length of the orbit

τ – unit vector tangential to the orbit

ϕ – magnetic flux

φ – phase of the particle with respect to the field under consideration

φ_s – symbol of stable-phase angle

$\psi(\vartheta)$ – periodic function describing the instantaneous closed orbit

ω(rad/sec) – frequency of revolution of the particle, frequency of the photon (ch. 5)

ω_0(rad/sec) – angular frequency of the variation of the accelerating field considered

Ω(rad/sec) – angular frequency of linear synchrotron oscillations

COMPARISON OF ACCELERATOR NAMES AND RELATED TERMS IN THE RUSSIAN AND ENGLISH SPEAKING COUNTRIES

Russian	*English*
electrostatic generator	electrostatic accelerator
linear accelerator	linear accelerator
cyclotron	cyclotrón
betatron	betatron
phasotron*	synchrocyclotron
synchrophasotron**	proton synchrotron
synchrotron***	electron synchrotron
ring-phasotron	FFAG (fixed-field alternating gradient) accelerator
symmetrical ring-phasotron	2-way FFAG accelerator
unsymmetrical ring-phasotron	1-way FFAG accelerator
microtron	microtron
stochatron	stochastic accelerator
storage system	storage ring(s)
coefficient of orbit expansion	momentum compaction factor
critical energy	transition energy

* Accelerator with constant magnetic field and variable r.f. frequency.
** Accelerator with variable magnetic field and variable r.f. frequency.
*** Accelerator with variable magnetic field and constant r.f. frequency.

CHAPTER 1

INTRODUCTION

1. ROLE OF ACCELERATORS IN NUCLEAR AND ELEMENTARY PARTICLE PHYSICS

Accelerators of charged particles are devices in which a high kinetic energy is given to the particles, in some cases up to several GeV* (for electrons) and even several tens of GeV (for protons). How significant these quantities are will be seen from table 1, which presents some characteristic energies of particles for comparison. It is evident that the energies of particles in large contemporary accelerators may well exceed the mean energy of cosmic rays.

In all accelerators, energy is given directly to various charged particles. At the end of acceleration one can obtain, with the help of various physical processes, secondary beams of charged particles (mesons, antiprotons, etc.) and also of neutral particles. For instance, accelerated electrons which are stopped in a target give rise to hard gamma radiation; accelerated protons, passing through matter, charge-exchange to produce fast neutrons; high energy deuterons penetrating special targets may dissociate on account of the 'stripping' reaction and give two beams of fast particles: neutrons and protons. Consequently, it is possible to get different kinds of high energy particles as reaction products from accelerators–electrons, protons, mesons, deuterons, alpha-particles, neutrons, hard photons and others. Also the interest in acceleration of multicharged ions (lithium, oxygen, etc.) has grown in recent years with a view to making transuranic elements and studying their properties.

The role of accelerators as instruments of experimental investigation in contemporary physics is extremely great, as the development of

* 1 GeV – one billion (milliard) electron volt – 10^9 eV = 1.6×10^{-3} erg.
1 MeV – one million electron volt – 10^6 eV = 1.6×10^{-6} erg.

nuclear and elementary particle physics is basically tied to the study of the interactions of the various high energy particles with atomic nuclei and between themselves, many problems related to these phenomena can be studied with accelerators in a much more efficient manner and under much more satisfying conditions than with the help of cosmic rays.

TABLE 1
A few characteristic values for the energies of particles

1.	Rest energy of the electron	$E_0 = 0.511$ MeV
2.	Threshold energy for the production of an electron-positron pair by gamma-rays	$E_{th} \gtrsim 1$ MeV
3.	Kinetic energy of alpha-particles from radioactive substances	$W \approx 5$–10 MeV
4.	Maximum kinetic energy of beta-rays (electrons) from radioactive substances	up to a few MeV
5.	Mean binding energy of the nucleon in the nucleus	$W_B \approx 8$ MeV
6.	Energy of the electron, for which its de Broglie wave-length is equal to the classical radius of the electron	$E = 137\, m_0 c^2 \approx 70$ MeV
7.	Rest energy of the π-meson	$E_0 \approx 140$ MeV
8.	Rest energy of the K-meson	$E_0 \approx 500$ MeV
9.	Rest energy of the proton	$E_0 \approx 938$ MeV
10.	Threshold energy for the production of a pair of K-mesons on a nucleon by gamma-rays	$W_{th} \approx 1.5$ GeV
11.	Threshold energy for the production of a pair of nucleons on a nucleon by gamma-rays	$E_{th} \approx 3.8$ GeV
12.	Threshold kinetic energy for nucleon pair production in the collision of a moving nucleon against one which is at rest	$W_{th} \approx 5.6$ GeV
13.	Mean energy of the primary component of cosmic rays	a few GeV
14.	Maximum energy of particles observed in cosmic rays	$E \approx 10^{17}$ eV

A large number of varied nuclear reactions studied with accelerators has allowed us to widen and to improve our representation of the atomic nucleus. Investigations on artificial radioactivity, which have been going on already for a long time with the help of accelerators, have produced rich experimental data on the energy levels of the halflives of their decay, the beta-spectra, etc. Recently many nuclear transformations have been

studied which were induced by hard gamma-rays produced with accelerators (nuclear photo-effect, photo-fission, etc.). Investigations on the so-called stars are now in progress – a phenomenon in which the nucleus is split into several fragments by a high energy particle.

The main significance of accelerators rests however with the fact that they provide the means of solving the most important and intricate problems of contemporary physics, concerning the nature, the interaction and mutual transformation of nuclear particles, and in particular the problem of nuclear forces.

Thus it was with protons from accelerators that antiprotons and antineutrons were produced and studied for the first time – particles which had been predicted theoretically for a long time.

According to present day concepts, π-mesons and perhaps also heavier mesons are the quanta of the nuclear field and play an essential role in nuclear forces. For this reason a decisive importance can be assigned to the work performed with accelerators to determine the basic characteristics of mesons (charge, mass, spin, etc.) and their decay properties, as well as to clarify the nature of the interaction of mesons with nucleons and between themselves. It is known that π-mesons were obtained with accelerators already long ago, in particular neutral π-mesons, by the interaction of fast nucleons between themselves as also by the interaction of hard gamma-rays with nucleons (the so-called photo-production of mesons). The study of the quantitative characteristics of meson production processes gives in some cases the possibility of chosing between various theoretical speculations.

The study of the so-called strange particles presents a particular interest. Not long ago, for instance, one of these strange particles called the antisigma-minus hyperon $\bar{\Sigma}^-$ was discovered in a large Soviet accelerator, the synchrophasotron of the Joint Institute for Nuclear Research in Dubna.

A particulary large amount of information is expected from the accelerators concerning the structure of elementary particles, for instance, the electrical structure of the proton. The experiments that are being prepared now to determine the structure of the electron and the limits of contemporary electrodynamics will have fundamental importance for the development of theory.

It has been possible recently, thanks to significant progress in acceler-

ator techniques, to produce by means of targets beams of secondary particles sufficiently intense for further experiments, i.e. beams of pions K-mesons and antiprotons.

Besides their value to science in the widest sense of the word, accelerators have also a not unimportant practical significance. We are thinking here, in particular, of their application to the preparation of artificially radioactive materials for research in medicine, biology and chemistry (by the method of labelled atoms). We mention also the use of gamma-rays from electron accelerators (in particular from betatrons) for the detection of defects in large masses of metal and for medical X-radiography. Also in medicine we find the application of hard X-rays and fast electrons to cancer therapy. There are also a few specialised applications such as the use of fast electrons for sterilizing food.

The branches of science in which high energy particles from accelerators find application continue to increase. We find also an increasing number of papers in which are investigated processes induced in various media by bombardment with fast particles, from the points of view of chemical transformation, change of the properties of semi-conductors or constitution of living tissue.

However, the basic application of accelerators is evidently for research in nuclear and elementary particle physics, where a new range of phenomena and laws is opened to investigation with each new order of magnitude of energy which it is possible to achieve. For this reason, the development and improvement of accelerators is stimulated by the requirements which at each stage present interest for further development in the various branches of physics. At the same time, the fundamental progress which has been achieved in the past 10 to 15 years in the construction of accelerators was made possible only by the latest achievements in physics and technology. An impression of this progress is given in the next section, together with a short history of accelerator development.

2. SHORT REVIEW OF THE DEVELOPMENT OF CYCLICAL ACCELERATORS

This short review has no pretention of being complete and is only intended to show the basic trends and future aspects of the development of accelerators.

The first attempts to use particles of high energy for physics purposes belong to Rutherford. This (his famous experiment, performed in 1919) consisted in splitting nitrogen nuclei with α particles, coming out of a radioactive preparation. The next important step forward was made in 1932 by Cockroft and Walton, who provoked a nuclear reaction for the first time with the help of particles accelerated in a special high voltage device (spallation of the lithium nucleus by 700 keV protons). Analogous experiments were soon performed also by Soviet physicists in the Kharkov physicotechnical institute. In the equipment used in those days, the energy was given to the particles in one single traversal of a large potential difference applied to the accelerating gap. This method (in its concept the most simple) is sometimes called the direct method of accelerating particles.

This direct method has found its final development in the electrostatic generator of Van de Graaf (1931).

In this device the large potential difference is obtained from electrostatic charge stored on the outer surface of a conducting sphere of large radius with the help of an endless belt. Modern electrostatic generators are reliable machines giving intense and well focussed beams of particles with energies of a few MeV and little energy spread. A significant increase of the voltage limit is hardly possible in view of the difficulties related to the constantly applied high tension and the danger of breaking down the insulation.

It is natural that already long ago attempts were made to find a means of obtaining particles of high energy which would not necessitate the use of very high potentials. To realise such 'indirect' methods of acceleration, systems of two different types were proposed: in the first kind the particles move in a straight line, passing once through each accelerating gap in the process of acceleration (linear accelerators); in the second the particles move in a circle or a spiral, accomplishing a cyclical motion and passing many times through the same accelerating gaps (cyclic accelerators).

Linear accelerators, proposed as early as 1924 by the Swedish scientist Ising, were designed and constructed for the first time by the Swiss physicist Wideröe (1928) and by Lawrence and Sloan (U.S.A. 1931). In these devices, based on the principle of resonant acceleration, the motion of the particles was synchronous with the change of the alternating

electric accelerating field; the period of variation of the electric field was equal to the time of flight of the particles between adjacent accelerating gaps. The non-resonance induction variant of the linear accelerator was proposed in 1928 by the German engineer Bauers.

In cyclic accelerators the particles, performing hundreds, thousands and even hundreds of thousands of revolutions cross each time the same accelerating gaps. To obtain a trajectory of circular form, a magnetic field of sufficiently high intensity is applied. For this reason the main part of any circular accelerator is a magnet of some form.

The first cyclic accelerator, the well-known cyclotron, was developed in the years 1930–1931 by Lawrence and his co-workers [1] at the University of California. The first in Europe, the cyclotron of the Radio Frequency Institute in Leningrad [2] was operated in 1937. Cyclotrons are resonance accelerators; the time of one revolution in the magnetic field must be equal to the period of oscillation of the high frequency accelerating electrical field existing between two special electrodes (the dees). Cyclotrons, which in former times played a great role in the development of nuclear physics, have not so far lost their importance. However, the cyclic resonance accelerators whose first representative was the cyclotron, later underwent considerable development and appear to be dominant over other types of accelerators.

Cyclotrons were and are used for the acceleration of heavy particles, protons, deuterons, α-particles, as a rule, up to 10–20 MeV. In recent years cyclotrons were also designed specially for the acceleration of multicharged ions (carbon, nitrogen, etc.). To obtain particle energies considerably higher than these figures it is necessary to increase the amplitude of the accelerating electric field between the dees. Physically this requirement is tied to the relativistic increase in mass of the accelerated ions and the subsequent decrease of their revolution frequency in the magnetic field of the cyclotron. This can be to a certain extent compensated by a considerable increase of the voltage amplitude. On this basis the construction of a big cyclotron with a weight of about 4000 t was started in 1939 at the University of California in order to produce deuterons with an energy of about 100 MeV. For this a high frequency voltage with an amplitude around 1–2 million Volt was required. One can conclude from this that for energies in the 100 MeV range the possibilities of the cyclotrons of the usual type are exhausted.

Another type of cyclic accelerator which has played an important part particulary in the early stages of their development, is the induction accelerator or, as it is often called, the betatron. This, unlike the cyclotron, is used for accelerating electrons. The motion of the electrons in the betatron under the action of the magnetic field takes place along trajectories which are nearly circular, as in the other cyclic accelerators. The basic difference is that betatrons are not resonance accelerators. The acceleration of particles is accomplished through the rotational electric field induced by the variation in time of the magnetic flux which links the orbit.

In 1928 Wideröe designed and constructed the first betatron, which he strangely enough was unable to put into operation. In the following years a series of other attempts to realise an induction accelerator were undertaken. It was only in the years 1940–1941 that Kerst in the U.S.A. was able to build a reliable accelerator of this type for an energy of 2–3 MeV. Nowadays betatrons, especially in the energy range 15–30 MeV, are widely used as electron accelerators, and are built on a production-line basis in some countries. In designing betatrons for energies of the order of 100 MeV and more, one meets severe difficulties. Apart from the increase in size of the magnet and especially of its core, the difficulties are related to the losses due to the electromagnetic radiation emitted by relativistic electrons at high energies.

As we have seen, a significant increase in the energy obtained with accelerators could not be attained either with cyclotrons (for heavy particles) or with betatrons (for electrons). For further progress with accelerators it was necessary to find new methods. These methods were invented in 1944 by the Soviet physicist Veksler [5] who proposed a series of new types of accelerators based on the principle of phase focusing which he had discovered. A similar proposal was made independently, but about one year later, by the American physicist MacMillan [6]. The essential of Veksler's proposal consisted in abandoning the previously accepted requirement of having an exact resonance (synchronism) between the particle revolution frequency and the frequency of the accelerating electrical field. It appeared that it was possible under certain not too stringent conditions to sustain automatically a regime of particle motion such that the energy increases gradually while performing oscillations around the value corresponding to exact reso-

nance. The discovery of the principle of phase focusing made it possible to design, and afterwards to build, a large number of accelerators of various types:

1. phasotrons (synchrocyclotrons) – accelerators with a magnetic field of the cyclotron type constant in time, and frequency of accelerating voltage varying with time,
2. synchrophasotrons (proton synchrotrons) – ring-shaped accelerators with acceleration frequency and magnetic field both varying with time,
3. synchrotrons – accelerators with magnetic field varying with time and a constant acceleration frequency.

Phasotrons and synchrophasotrons are used for accelerating heavy particles, primarily protons, and synchrotrons for accelerating electrons.

With the invention of these new machines a new stage began in the development of accelerators, which made it possible to raise the energy of the accelerated particles by 2 or 3 orders of magnitude. Beginning in 1945, a series of synchrotrons (for energies of 300–500 MeV) and phasotrons (for energies of 400–600 MeV) were built in a short time. Of these a synchrotron for 270 MeV (Lebedev Institute of the Academy of Sciences, Moscow) and a phasotron for 680 MeV (Joint Institute for Nuclear Research, Dubna) were in the U.S.S.R. After this began the construction of enormous accelerating machines (proton synchrophasotrons) designed for energies of the order of several milliards of electron volts (several GeV), i.e. corresponding to the mean energy of cosmic rays. These devices were built and put into operation in the course of a few years: in 1952 the Cosmotron of 3 BeV (Brookhaven, U.S.A.), in 1954 the bevatron of 6.2 BeV (Berkeley, U.S.A.), in 1957 the Soviet synchrophasotron of 10 BeV (J.I.N.R., Dubna).

Parallel with the development of cyclic accelerators the improvement and development of linear accelerators went forward. In the pre-war years no high figures of particles energy had been attained. Only in the post-war years did it become possible to raise the efficiency of linacs significantly, on the basis of the phase focusing principle and the achievements of microwave techniques.

Linacs were then constructed using modern resonators, wave-guides and powerful r.f. sources at microwave frequencies. Albeit high-energy linacs did not receive such a wide diffusion as the cyclical accelerators

had had. At present there is only one accelerator of electrons for 700 MeV (Stanford, U.S.A.) and a series of proton linacs for energies of a few tens of MeV which are used as a rule as injectors into the big synchrophasotrons.

Around the years 1952–1953 a new principle began to make headway, known under the name of strong focusing. As we have recalled, the large synchrophasotrons give proton energies up to 10 GeV. The question arose of what possibilities there were to obtain still higher energies. Was it possible, in order to obtain protons with an energy of tens of GeV, to build accelerators similar to the synchrophasotrons just described, only of larger dimensions? Already the American 6 GeV Bevatron and the Soviet 10 GeV accelerator in Dubna had weights of 10000 and 36000 tons respectively, and needed a power supply of extremely high rating. It was imperative, in order to reduce the weight of the magnet and the power required, to be able to reduce the orthogonal dimensions of the magnet gap (aperture) without significant loss of beam intensity.

Such a decrease in aperture is tolerable only on condition that the focusing forces orthogonal to the orbit, which maintain the particles inside the narrow annular region between the magnet poles, are increased. Precisely such an increase of the transverse focusing forces is obtained in strong focusing systems, the idea of which was proposed in 1950 by the Greek engineer Christofilos [7]. However, strong focusing accelerators began to be developed after the publication, at the end of 1952, of the work of Courant, Livingston and Snyder [8] at Brookhaven, U.S.A., who discovered once more this effect and published preliminary parameters for large strong focusing proton accelerators. Soon afterwards there began in various countries the design and construction of large proton accelerators of this type, of which there are now two in operation at energies of about 30 GeV, one at the European Organisation for Nuclear Research (CERN) operated by representatives of 13 Western European nations at Geneva [9], and the second at Brookhaven National Laboratory, U.S.A. [10]. Both of these accelerators have already been put into service, the first in 1959, the second in 1960. In the Soviet Union a synchrophasotron of 7 GeV has been built, and the construction of the biggest proton accelerator, for an energy of 70 GeV, is in progress [11]. Other strong focusing accelerators are being designed or constructed, for instance for the acceleration of electrons to energies of 5–6 GeV.

We can remark that the mechanism of acceleration with strong focusing, although displaying several peculiarities, does not differ in principle from the corresponding mechanism in weak focusing machines (phase focusing, etc.). It is possible to use strong focusing also in linear accelerators.

Two basic trends are found in acceleration development. The first is to raise the attainable energy E_{max}, the second, to raise the intensity I of the current of accelerated particles. The experience of the last decade has shown that the advance towards accelerators of higher energies E_{max} is usually accompanied by a decrease of the intensity I. This leads to the situation that experimental possibilities with large accelerators are severely limited, that the time needed for an experiment gets longer, that the precision decreases, and so on. For this reason a large amount of work has been done recently on ways of increasing the intensity I as well as on various technical improvements of the properties of the injectors, (increase in the precision of bunching, and so on) and on new methods of acceleration, allowing in principle a substantial increase in beam intensity.

As an example we give the values of the mean current of accelerated particles $\bar{I}$ and their number per second $\bar{n}$, for accelerators of the various types at present in operation:

Cyclotrons:	$\bar{I} \sim 0.1$–1 mA,	$\bar{n} \sim 10^{15}$	proton/sec
Phasotrons:	1 μA,	10^{12}–10^{13}	proton/sec
Synchrophasotrons:	10^{-4}–10^{-2} μA,	10^{9} –10^{11}	proton sec
Synchrotrons:	10^{-4}–10^{-2} μA,	10^{9} –10^{11}	electron/sec

The greater mean intensity in cyclotrons is basically due to the fact that the capture of the particles into the acceleration regime and the forming of the beam of accelerated particles occur in these machines practically without interruption. In the other devices mentioned, which display the property of phase focusing, the accelerated beam is obtained only during short periods of time, separated by large intervals, i.e. in pulses. The basic defect of the cyclotron – its inadequacy in accelerating relativistic particles – can be overcome up to a certain degree in the so-called isochronous cyclotron, the concept of which was demonstrated already in 1938 by Thomas [12]. In recent years the first relatively small machines of this type have been designed and built [13, 14].

The first proposal of the strong focusing principle referred to machines with a time-varying magnetic field. However its application to accelerators with magnetic fields constant in time not only presents definite advantages from the engineering point of view, but also makes it possible to increase substantially the efficiency of the device, i.e. to increase the intensity of the accelerated beam. In 1953 the Soviet physicists Kolomensky, Petukhov and Rabinovitch [15] proposed a new type of strong focusing accelerator with a magnetic field constant in time but with a considerable variation along the radius (the annular phasotron). In 1955 analogous proposals were made by Symon, Kerst and their co-workers [16].

From what we have said about focusing and the acceleration methods in cyclic accelerators, it is possible to divide them into the groups shown in table 2. Of all the accelerators listed in this table, the weak-focusing

TABLE 2
The different forms of cyclic accelerators

Magnetic guiding field applied	Type of focusing	
	weak	*strong*
Field constant as function of time	cyclotron, phasotron (synchrocyclotron), microtron, weak-focusing stochatron.	ring-phasotron of radial or spiral type with resonance or induction acceleration, cyclotron with azimuthal or spiral variation of the magnetic field, strong-focusing stochatron.
Field varying with time	weak-focusing synchrotron, synchrophasotron, betatron.	strong-focusing synchrotron, synchrophasotron, betatron.

cyclotron, the phasotron, the synchrotron, the synchrophasotron, the betatron and the strong-focusing synchrophasotron have had the widest application.

In recent years machines have been devised to realise the collision of intense beams of relativistic particles travelling in opposite directions [17], (see also § 3). In this case the centre of mass of the system remains

at rest and the energy available for useful physics purposes, for instance the generation of new particles, is much greater than that obtained in the usual fashion, when the accelerated beam hits a stationary target. The practical realisation of the colliding beams method requires a substantial increase of the intensity of the accelerated beams. To this effect the strong focusing accelerators with time-independent magnetic field can be used, also special storage magnets where one can stack the current of particles accelerated in a separate high current machine (cyclic or linear). The construction of such a storage system for 500 MeV electrons provided by a linear accelerator is being completed at Stanford, U.S.A. [18].

The machines with a time independent magnetic field we referred to can be, in particular, 'symmetrical' accelerators [19], (see also ch. 7, § 3), where it is possible to accelerate and stack at the same time beams of particles circulating in opposite directions and to realise their head-on collision. The first accelerators of this type have been already developed and built. Besides the work on colliding beam machines, design and development work is going on in various countries with the aim of realising extremely large proton accelerators for energies of a few hundred GeV. Of all the existing types only the strong focusing synchrophasotron can be adopted for the construction of such large accelerators, the radius of which will be of the order of a kilometre. With such dimensions, it is necessary to impose extremely severe tolerances for the machining, construction and assembly of the machine. For example, the precision in setting up the magnet blocks is given by fractions of a millimetre. Such requirements are on the limit of technical possibilities and it is necessary to look carefully for ways of ensuring correct and reliable performance of these big machines.

One of these ways, which is gaining importance in recent years, is the use of 'self-regulation'. Deviations of the beam from the ideal motion are registered by special devices and the received signals are used to correct the appropriate parameters. This method of self-regulation of the accelerator using the beam itself, which has justified itself in practice, already increases the reliability and efficiency of the accelerators and makes it possible to relax the tolerance requirements.

Another trend in the planning of very large accelerators, is towards the design of 'cascades', consisting of two ring-shaped cyclic accelerators for

example. The particles are first accelerated in a small ring up to an intermediate (usually not very high) energy, and then transfered to a large ring where acceleration to the final energy takes place. The advantage of such a system lies in the fact that the aperture of the second larger ring can be made relatively small as the amplitudes of the transverse oscillations have been damped down during the acceleration process in the first ring. The realisation of cascade systems presents some difficulties related to the requirement of efficiently removing the beam from the first ring and injecting it into the second.

The rapid development of the various types of accelerators has required the solution of many theoretical questions, to which a large number of authors contributed at various stages. Among them we can mention the Soviet physicists Terletzky, Vladiminsky, Rabinovitch, Dmitrievsky, Sokolov and their collaborators, as well as the American scientists Bohm, Foldy, Courant, Symon, Sands, etc.

Apart from the methods making use of various combinations of electric and magnetic fields to guide and accelerate the particles, investigations have been made recently on new methods of focusing and acceleration with the help of plasmas of electrons and ions. The concepts of these methods, which distinguish themselves by their variety, have been given and are being developed by the Soviet physicists Budker [20], Veksler [21], Fainberg [22] and their collaborators. Up to now it is not clear what practical future plasma methods of acceleration will have, because, although they are very efficient, they are dependent on the resolution of very great difficulties. We shall not describe these methods in our book as they are at present only in their initial stage of development.

At the end of this short review and before we go over to the theory of the motion of particles in accelerators, it is appropriate to give a general description of the construction and functioning of a modern cyclic accelerator. We shall do this using as an example one of the largest accelerators, the Soviet synchrophasotron of the Joint Institute of Nuclear Research (J.I.N.R.), [23], which is situated in the town of Dubna.

The main part of the accelerator is the ring-shaped electromagnet, consisting of four sectors with a mean radius of curvature $R = 28$ m, separated by four 8 m long straight sections (fig. 1). Between the magnet pole pieces, where the guiding field is produced, a vacuum chamber is

placed in which the particles (in this case protons) travel on nearly circular orbits under the influence of the field. The chamber is evacuated down to a working pressure of $3–15 \times 10^{-6}$ mm Hg.

The protons, coming from an ion source, are first accelerated in a cascade generator (up to 600 KeV) and then in a linear accelerator

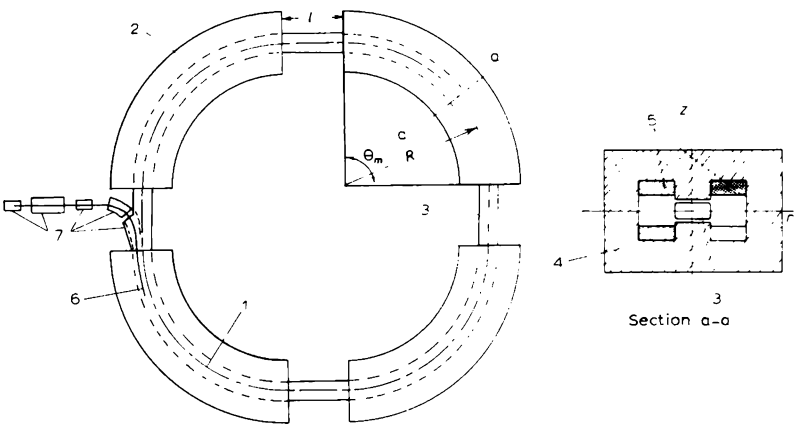

Fig. 1. Layout of the weak-focusing 10 GeV synchrophasotron of the Joint Institute for Nuclear Studies.

1. basic orbit; 2. sector (quadrant) of the electromagnet; 3. vacuum chamber; 4. magnet yoke; 5. magnet coils; 6. beam of particles; 7. injector-accelerator and inflector.

(up to 9 MeV). After this the proton beam passes through an intricate bending and focusing system which is situated in one of the straight sections, and enters the vacuum chamber of the accelerator. At this moment the accelerating electric field is not yet turned on and the particles are stored on orbits which, on account of the increase of the magnetic field with time, are stacked side by side in the chamber, on the side of the decreasing radii. The injection of the beam goes on for a time corresponding to several tens of revolutions, after which the high frequency accelerating voltage is applied to electrodes placed in two diametrically situated straight sections. The particles begin to be accelerated in the resonance regime, which lasts 3 sec. At the end of the acceleration period the magnetic field on the orbit has grown to its maximum value $B_{max} \approx 13000$ Gauss, and the energy of the protons reaches the value $W_{max} \approx 10$ BeV.

The protons, accelerated to W_{max}, are directed towards various targets where they generate the secondary particles which are the object of the physics experiments. The fourth straight section serves for the extraction of these particles towards the experimental hall.

The magnetic field, after having reached B_{max}, falls to zero in a time about equal to the rise-time. After this follows a period of rest of the same length. Then the field begins to rise again and the cycle is repeated (fig. 2). In this way, every 10 12 sec, the accelerator delivers a beam containing 10^{10} protons at an energy of 10 GeV.

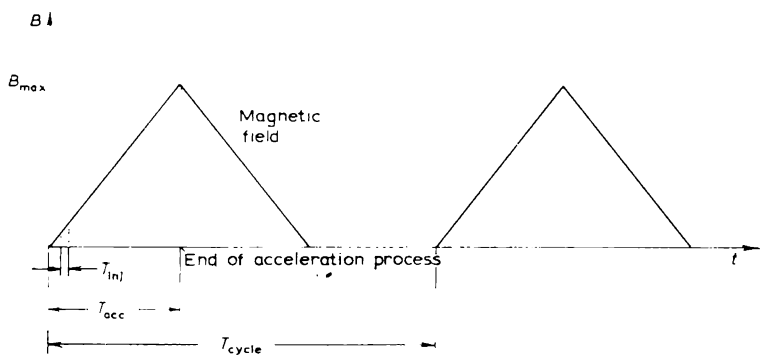

Fig. 2. Graph of the variation of the magnetic field with time in the synchrophasotron (schematic).

During the major part of the acceleration process the protons move with a velocity near to that of light and travel in 3 sec a distance of about 900,000 km, corresponding to 4.5 million revolutions on the orbit.

In order to avoid dangerous perturbations of the normal modes of motion and acceleration of the particles on such a long path, the magnet and its power supply, the high frequency accelerating system, the elements of the injection system and the other parts of the machine must be prepared to high technical standards with extremely great precision, and be equipped with a series of regulating, controlling and correcting devices. Also the operation of all the parts has to be synchronized with the same high degree of accuracy.

3. A FEW ENERGY RELATIONSHIPS

The choice of the method of acceleration and of the type of the accelerator depends, as one would aspect, on the particle in question which can be heavy, as protons, deuterons, etc., or light as electrons. The latter, because of their small mass, attain a very high velocity in an extremely short time. For instance, already at an energy of 1 eV, the electron will have a velocity of about $v \approx 600$ km/sec, corresponding to $\beta = 0.002$ where β is the ratio v/c. At an energy of 1 MeV, the velocity is already near to that of light c ($\beta = 0.94$). This is why the electron can be taken as relativistic in an accelerator, i.e. it will be moving along with a practically constant velocity $v \approx c$, whereas its mass will increase many times during the acceleration process. A heavy particle, such as a proton, is still very far from being relativistic at an energy of 1 MeV ($\beta \approx 0.045$), and will only become so in the energy range above 1 GeV. As a guide fig. 3 gives the variation of velocity and mass for electrons and protons as a function of their kinetic energy.

The most important parameters of any cyclic accelerator are the radius curvature R of the orbit and the value of the strength of the magnetic field B. The kinetic energy W of the particle which can be kept on this orbit may be expressed very simply with the help of these two parameters: $W = W(BR)$. To this effect we make use of the condition

$$\frac{mv^2}{R} = \frac{Ze}{c} vB\,, \tag{3.1}$$

where m is the total mass of the particle and Ze its charge. We shall also use the formula of relativistic mechanics which links the kinetic energy and the momentum of the particle

$$W = \sqrt{c^2 p^2 + E_0^2} - E_0\,, \tag{3.2}$$

E_0 being here the rest energy and p the momentum of the particle. With the help of these formulae one finds the relationship between W and BR, as well as between p and BR:

$$W = \sqrt{(ZeBR)^2 + E_0^2} - E_0\,, \qquad p = \frac{Ze}{c} BR\,. \tag{3.3}$$

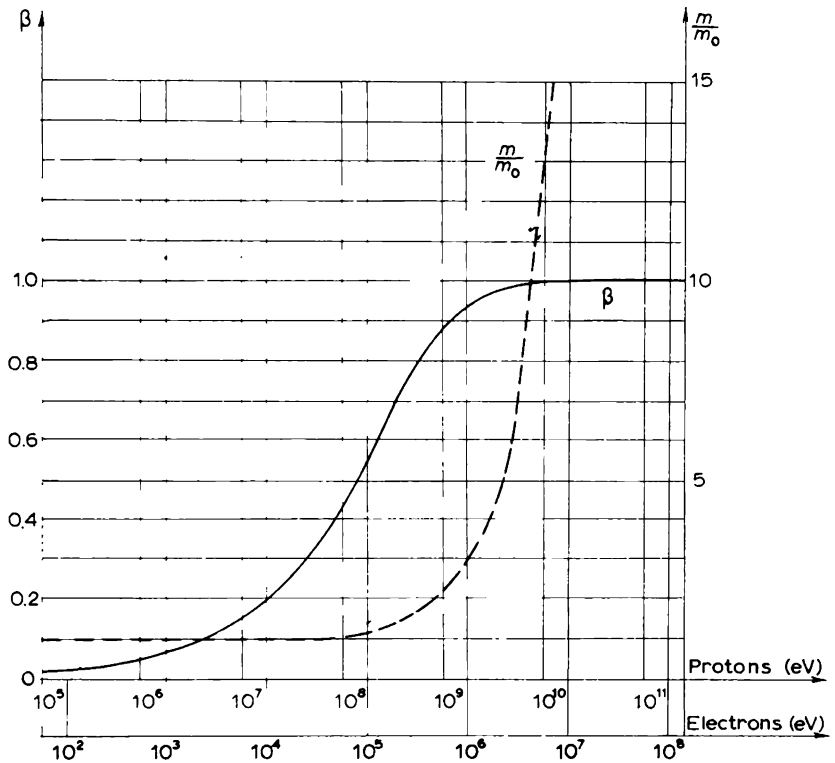

Fig. 3. Graph of the variation of the velocity and mass of electrons and protons as a function of their kinetic energy.

From here follow formulae which are approximated, but useful for computations, for the non-relativistic case ($\beta \ll 1$)

$$W \approx 4.5 \times 10^4 \frac{(ZBR)^2}{E_0} \tag{3.4}$$

and for the relativistic case ($\beta \approx 1$)

$$W \approx pc = 300\, ZBR\,, \tag{3.5}$$

where W and E_0 are expressed in electronvolt, B in Gauss, R in centimetres.

We indicate some values of the quantities W_{max}, B_{max} or R_{max} for accelerators of various types:

a) cyclotron (protons): $W_{max} \approx 10$ MeV, $B \approx 18$ kG, $R_{max} \approx 25$ cm.
b) phasotron (protons): $W_{max} \approx 700$ MeV, $B \approx 16$ kG, $R_{max} \approx 2{,}8$ m.
c) synchrophasotron (protons): $W_{max} \approx 25$ GeV, $B_{max} \approx 13$ kG, $R \approx 65$ m.
d) betatron (electrons): $W_{max} \approx 50$ MeV, $B_{max} \approx 6$ kG, $R \approx 28$ cm.
e) synchrotron (electrons): $W_{max} \approx 350$ MeV, $B_{max} \approx 12$ kG, $R \approx 100$ cm.
f) synchrotron (electrons): $W_{max} \approx 1.2$ GeV, $B_{max} \approx 13$ kG, $R \approx 3$ m.

In case a) one can use eq. (3.4), in cases c) to f) eq. (3.5), while in case b) it is necessary to use the complete formula (3.3).

It is necessary to stress that it is important from the point of view of experiments with accelerators to know the value of the energy in the centre of mass of the accelerated particle and the particle with which it has an interaction. Let us look into this question in a little more detail.

Let us assume two particles moving in the laboratory system, with rest masses, total energies and momentum respectively equal to m_{01}, E'_1, p'_1 and m_{02}, E'_2, p'_2. From relativistic mechanics (see for instance ref. 24), it is known that the expression $E^2 - \boldsymbol{p}^2 c^2$, where E is the total energy and $\boldsymbol{p}$ the momentum of the system, is invariant (it is the square of the length of the energy-momentum four-vector, or the square of the rest energy), the value of which remains the same in all inertial systems of co-ordinates.

$$E^2 - \boldsymbol{p}^2 c^2 = \text{inv.} \tag{3.6}$$

Making use of this formula, and of the fact that $\boldsymbol{p} = 0$ in the centre of mass system by hypothesis, we may write the equation

$$(E'_1 + E'_2)^2 - (\boldsymbol{p}'_1 + \boldsymbol{p}'_2)^2 c^2 = (E_1 + E_2)^2 , \tag{3.7}$$

where primes refer to the laboratory system of co-ordinates and quantities without primes are taken in the centre of mass system.

Now, keeping in mind that for each of the particles we have

$$E_i'^2 = (m_{0i}c^2)^2 + p_i'^2 c^2 \qquad (i = 1, 2)\,, \tag{3.8}$$

we obtain for the energy of both particles in the centre of mass system

$$E = E_1 + E_2 = \{(m_{01}c^2)^2 + (m_{02}c^2)^2 + 2(E_1' E_2' - \boldsymbol{p}_1' \boldsymbol{p}_2' c^2)\}^{\frac{1}{2}}\,. \tag{3.9}$$

The same expression can be obtained in another way, by using the relativistic representation of the velocity when going over to the centre of mass system.

Let us consider, to begin with, the case usually encountered with accelerators, where the particle of the accelerated beam, with the full energy E_1', impinges on a target particle at rest. In this case one has to set

$$\boldsymbol{p}_2' = 0\,, \qquad E_2' = m_{02}c^2\,, \tag{3.10}$$

and eq. (3.9) becomes

$$E = \{(m_{01}c^2)^2 + m_{02}c^2(m_{02}c^2 + 2E_1')\}^{\frac{1}{2}}\,. \tag{3.11}$$

Knowledge of the value of the energy available in the centre of mass system makes it possible to determine the thresholds of the reactions which can take place at the impact of the particle which is moving along on the particle which is at rest. Such a reaction can be, for instance, the production of new particles. Let us, as an example, determine the threshold kinetic energy W_{th}' of the moving particle which is required for the production of k particles with a rest mass m_{03}. To determine this threshold, we must consider that after the collision not only the colliding particles m_{01} and m_{02} but also all the k newly created particles are at rest in the centre of mass system. In accordance with (3.11) we get the expression

$$W_{\text{th}}' = E_{\text{th}}' - m_{01}c^2 = k\left(1 + \frac{m_{01} + \frac{k}{2}m_{03}}{m_{02}}\right)m_{03}c^2\,. \tag{3.12}$$

This expression shows that W_{th}' always exceeds the value $km_{03}c^2$ of the sum of the rest energies of all particles produced, and by an amount which can be quite considerable. The physical sense of this result is easy to understand, if one remembers that, when $W' = W_{\text{th}}'$, the particles

are at rest only in the centre of mass system. At the same time they are all moving together relative to the laboratory co-ordinate system with a velocity equal to that of the centre of mass. The amount of energy necessary to give this motion to the system of particles as a whole is then just equal to the surplus of energy present in W'_{th} over $km_{03}c^2$. If the energy of the incoming particle W' is greater than the threshold energy W'_{th}, then the particles will not be at rest any more even in the centre of mass system.

Let us consider a few simple particular cases.

1. If the result of the collision of an incoming particle with a particle at rest having the same mass is the production of a pair ($k = 2$) of new particles with masses equal to the original ones ($m_{01} = m_{02} = m_{03} = m_0$), then we shall have $W'_{th} = 6\,m_0c^2$ in accordance with (3.12). Thus, the threshold energy of a proton needed to produce a nucleon-antinucleon pair (proton-antiproton or neutron-antineutron) in a collision with a nucleon, is $W'_{th} \approx 5.6$ GeV.

2. If the irradiation of the target is carried out by gamma rays and if the masses of the particle at rest and of all created particles are the same, $m_{01} = 0$, $m_{02} = m_{03} = m_0$, then eq. (3.12) gives for the creation of a pair of particles the condition $E'_{th} = 4\,m_0c^2$. For instance the threshold energy of gamma rays for the production of a nucleon-antinucleon pair on a nucleon is equal to $E'_{th} = 3.75$ GeV. As we see, this is considerably less than the value W'_{th} for the creation of the same pair of particles by a proton.

3. If the incoming and the target particle have the same mass which is different from the mass of the single particle created $m_{01} = m_{02} = m_0 \neq m_{03}$, then (3.12) leads to the result $W'_{th} = (2 + (m_{03}/2m_{02}))m_{03}c^2$. Thus the production of one π-meson ($k = 1$) in a nucleon-nucleon collision has a threshold energy of $W' \approx 290$ MeV.

4. We have the general case $m_{01} \neq m_{02} \neq m_{03}$ for instance when gamma rays produce on a nucleon a pair of π-mesons (photoproduction of a pair of π-mesons). Then (3.12) gives $E'_{th} \approx 322$ MeV.

Let us stress once more that the production of particles has been considered here only from the point of view of the laws of conservation of mass and energy. On the basis of these laws it is only possible to say that the production of a given particle or group of particles is impossible, when the energy of the incoming particle is smaller than the threshold

$W_1' < W_{th}'$. The problem of the intensity of production (its effective cross-section) cannot be solved on the basis of the conservation laws and needs for its solution theoretical and experimental data on the nature of the interaction.

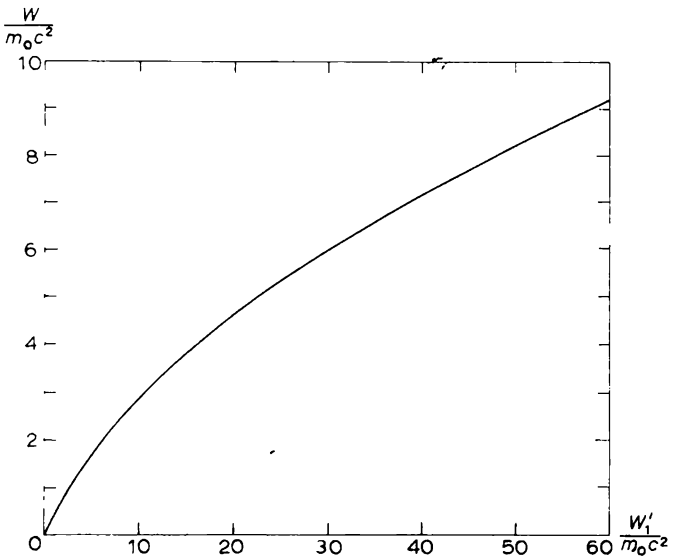

Fig. 4. Graph of the energy available for the experiment in the centre of mass system as a function of the kinetic energy of the incoming particle.

The considerations outlined above are illustrated in fig. 4, which shows for the case $m_{01} = m_{02} = m_0$ the dependence of the energy available for the experiment in the centre of mass system $W = E - 2m_0c^2$ on the kinetic energy of the incoming particle W_1' (in m_0c^2 units):

$$\frac{W}{m_0c^2} = 2\left(\sqrt{1 + \frac{W_1'}{2m_0c^2}} - 1\right). \tag{3.13}$$

As we see, the part of the energy available for the experiment W grows very slowly, something like $\sqrt{W_1'}$. For instance, in a proton phasotron of 10 GeV, the available energy in the case of a collision with a nucleon

is around 2.8 GeV. We find a still larger difference between W and W' in the case of the collision of extremely relativistic electrons with electrons at rest. For instance with energies of the incoming electrons of the order of $W_1' \approx 1000$ MeV, the energy in the centre of mass system amounts in all to $W \approx 30$ MeV.

The efficiency of use of the accelerated particles can be increased, in principle, by the method of colliding beams moving in opposite directions, so that their centre of mass remains stationary before and after the collision (in the simplest example). This idea was known for a long time, but its realisation seemed practically impossible, as the intensity of the accelerated beams was low and the number of encounters between particles of the colliding beams too small for experimental purposes. Only recently with the development of storage magnets and accelerators of various types delivering large currents has a real possibility appeared of obtaining circulating relativistic beams of high density, sufficient in principle for experiments (see § 2 and ch. 7 §§ 3, 4).

Let us have a look at the energy relationships characterizing the colliding beams method. Let identical particles move against each other in the laboratory co-ordinate system with energies E_1', E_2' and momenta $\boldsymbol{p}_1', \boldsymbol{p}_2'$ respectively

$$m_{01} = m_{02} = m_0 , \quad \boldsymbol{p}_1' = -\boldsymbol{p}_2' . \tag{3.14}$$

Under these conditions we obtain from the general formula (3.9) the following expression for the energy in the centre of mass system

$$E = \sqrt{2} m_0 c^2 \{1 + \gamma_1' \gamma_2' + (\gamma_1'^2 - 1)^{\frac{1}{2}} (\gamma_2'^2 - 1)^{\frac{1}{2}}\}^{\frac{1}{2}} , \tag{3.15}$$

where γ_1', γ_2' are the relativistic factors

$$\gamma_i' = \frac{E_i'}{m_0 c^2} = \frac{1}{\sqrt{(1-\beta_i^2)}} \qquad (i = 1, 2) . \tag{3.16}$$

If the energies of the particles are relativistic, i.e. $\gamma_1' \gg 1$, $\gamma_2' \gg 1$, then

$$E \approx 2\sqrt{\gamma_1' \gamma_2'} m_0 c^2 . \tag{3.17}$$

If also $E_1' = E_2' = E$, then it is evident that $E = 2E'$.

It is interesting to compute what 'equivalent' energy E_{eq}' a particle must have when impinging on a target at rest, in order to obtain the

same energetic effect (the same value of energy in the centre of mass system) as in the case of two colliding beams. Comparing eqs. (3.11) and (3.17) one finds that at relativistic energies E_{eq} is equal to

$$E'_{eq} \approx 2\gamma'_1\gamma'_2 m_0 c^2 . \tag{3.18}$$

As an example, to get an energy of $E \approx 20$ GeV in the centre of mass system which can be obtained by the collision of two 10 GeV proton beams ($\gamma' \approx 10$), a proton accelerator with a target at rest should be designed for an energy of 200 GeV.

The gain in the value of the equivalent energy is especially important for extreme relativistic energies, characteristic of electrons [25]. For instance, with $E'_1 \approx E'_2 \approx 500$ MeV ($\gamma' = 1000$), we obtain $E_{eq} = 1000$ GeV, that is a gain of 2000. It is of practical importance that electrons with very large γ' values can be obtained with relatively small machines, much smaller than the accelerators designed to obtain relativistic protons.

4. BASIC CONCEPTS IN THE THEORY OF ACCELERATORS

Alongside the growing complication and improvement of the accelerators, the theory of the motion of the particles in these machines was developed and specialized, as the problems varied greatly with the type and size of machine. The role of the theory is not limited at all to indicating new possibilities of raising the energy or the intensities of the machines in principle. These complicated and expensive pieces of equipment can only be designed and constructed on the basis of a careful study of all factors liable to have an influence on their performance. For this reason the main duty of the theory appears to be the study of the various phenomena which can perturb the magnetic field, the accelerating system and so on, as well as the evaluation of factors like the scattering of the particles in the residual gas in the vacuum chamber, the existence of electromagnetic radiation where the motion is taking place in a magnetic field, etc.

At the present time the list of questions which is the object of the theory of accelerators is rather well defined. We shall dwell briefly in this paragraph on the most basic of these questions, trying to give more a qualitative picture of the related phenomena, keeping the rigorous theory as the subject of later chapters.

4.1. *Stability of ideal orbits in accelerators*

As a rule it is possible in cyclic accelerators to indicate a physically distinguished trajectory (orbit), determined for instance by the condition of synchronism with the accelerating field. One can call this equilibrium motion. However, already at the beginning of the acceleration process, at the moment of injection, the particles of the beam acquire a spread in co-ordinates, angles and energies around the 'equilibrium' values. After this, the particles usually travel a long way in cyclic accelerators, accomplishing a large number of revolutions (up to a few millions). In the course of this motion the particles are unavoidably submitted to the action of various perturbations, resulting from defects of the magnetic systems, lack of precision in the accelerating devices, too poor a vacuum, etc. For these reasons the stability of particle motion has a decisive importance for the normal operation of an accelerator.

To ensure stability the magnetic guiding field must have in the neighbourhood of the equilibrium orbit a configuration which fulfils particular requirements. Depending on how stable the motion is in directions perpendicular to the orbit, or, as it is termed, depending on the focusing forces, it is possible to divide somewhat arbitrarily all accelerators into two types: those with weak and those with strong focusing.

Historically the first to be applied to accelerators and the simplest of all was the azimuthally-symmetrical magnetic field, which is characterized by the fact that all its properties are independent of the azimuthal angle θ. The equilibrium motion of the particles in such a field is evidently a circle of a radius R determined by eq. (3.3). The motion will be stable if, at the moment when there is a small displacement of the particle with respect to the equilibrium circle, a restoring force appears which gives rise to an oscillation around the circle in the radial (r) and vertical (z) directions (fig. 5).

Let us consider first the motion in the plane $z = 0$. To have radial stability it is necessary that the centripetal Lorentz force $F_L = (e/c)vB_z(r)$, resulting from the vertical component of the magnetic field B_z, fulfils the following conditions

$$\begin{aligned} &F_L < F_C \quad \text{for} \quad r < R\,, \\ &F_L > F_C \quad \text{for} \quad r > R\,, \end{aligned} \qquad F_C = \frac{mv^2}{r}\,. \tag{4.1}$$

For small radial displacements from the equilibrium orbit, we can write

$$F_C(r) = F_C(R)\left[1 - \frac{x}{R} + \ldots\right],$$
$$F_L(r) = F_L(R)\left[1 - n\frac{x}{R} + \ldots\right], \tag{4.2}$$

where $x = r - R$, and n is a quantity which is often encountered in the theory of accelerators, called the magnetic field index

$$n = -\frac{R}{B_z(R)}\left(\frac{\partial B_z}{\partial r}\right)_{r=R}. \tag{4.3}$$

This quantity characterizes the rate of change of the magnetic field in the neighbourhood of the equilibrium orbit.

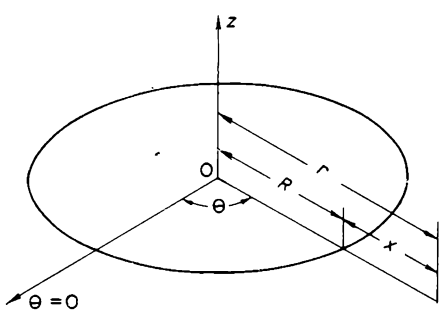

Fig. 5

From (4.1) and (4.2) the stability condition for the radial motion appears to be the inequality

$$n < 1, \tag{4.4}$$

which shows that the field should grow with the radius or at the limit not fall faster than $\sim 1/r$ (fig. 6).

Let us consider now the motion in the vertical direction (along the z-axis). The force acting in this direction, $F_x = (e/c)vB_x$, appears only if there exists a radial component of the magnetic field B_x. It is evident that in the symmetry plane of the magnetic field $z = 0$ where the equilibrium orbit lies, $B_x = 0$.

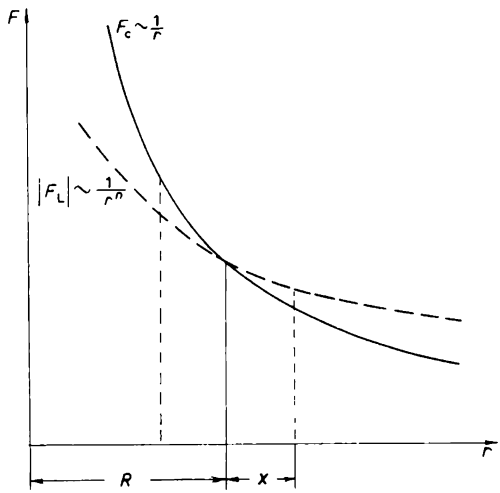

Fig. 6. Graph explaining the formation of a radial restoring force.

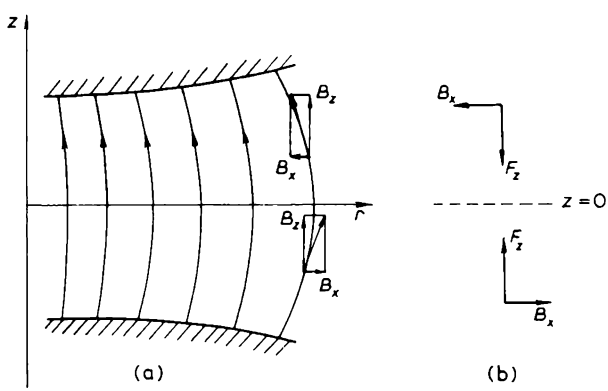

Fig. 7. (a) Example of the aspect of the lines of force of the magnetic field in the (r, z) plane; (b) vertical restoring force resulting from the action of B_x.

If B_z falls with radius ($(\partial B_z/\partial r) < 0$), the lines of force of the magnetic field will have the characteristic 'barrel' form and the force F_z will act on the particle in the direction of the z-plane, that is will be focusing (fig. 7). If on the contrary B_z rises with radius, this force will bring the

particles away from the plane $z = 0$, that is will be defocusing. The vertical stability condition can then be written with the help of the quantity n in the form

$$n > 0 . \tag{4.5}$$

In this way, the conditions of instantaneous stability in the radial and vertical directions in the azimuthally symmetrical field may be described by the following well known relation

$$0 < n < 1 . \tag{4.6}$$

The existence of quasielastic forces bringing back the particles to the orbit and proportional to the displacements x and z, causes the particle to perform oscillations known as 'betatron oscillations' around the orbit in the course of its motion. For sufficiently small displacements x and z these oscillations can be described in the azimuthally symmetric field by the following linearized equations [26, 27]:

$$\frac{d^2x}{dt^2} + \omega^2(1-n)x = 0 , \tag{4.7a}$$

$$\frac{d^2z}{dt^2} + \omega^2 nz = 0 . \tag{4.7b}$$

We notice that because of the condition (4.6) the frequencies of the betatron oscillations are less than the revolution frequency

$$\omega = (eB_z(R))/mc.$$

The origin of the term 'betatron' oscillations comes from the fact that the first investigations on orbit stability were made in detail on the induction accelerator (betatron), where incidentally there are no other characteristic oscillations of particles. The concept of betatron oscillations can be generalized in a natural way even for the case of much more intricate machines. In the literature one finds also the term 'free oscillations'. As we see it, this expression is much less appropriate as it can be applied with equal right to any other characteristic oscillation in an accelerator which can take place without the action of an external force.

The study of stability in the azimuthally symmetric field is particulary

simple and intuitive*. However the further development of accelerators has moved in the direction of substantial complication of the magnetic systems used and departure from azimuthal symmetry. This results in the first place from the fact that the frequency of betatron oscillations in the azimuthally symmetric field is always lower than the revolution

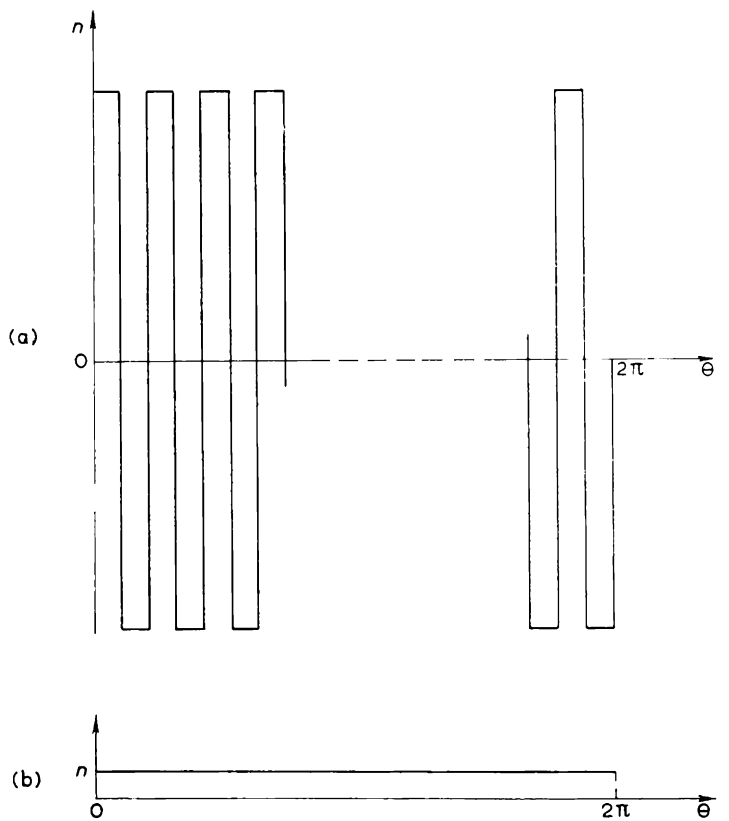

Fig. 8. Example of the dependence of the field index n on the azimuth: (a) for strong focusing, (b) for weak focusing.

* It is possible, in an azimuthally symmetric field not varying with time, to study in a general way the stability not only of small excursions from the equilibrium orbit but also of large ones (see appendix E).

frequency. At the same time, the lower these frequencies are (that is the weaker the focusing), the larger will be the excursions from the equilibrium orbit that are accomplished by the particles in the course of their motion under otherwise the same conditions. The trend to reduce these displacements whenever possible made it necessary to look for means of increasing the orbit stability, that is to find a so-called 'strong' focusing.

It happened that it was possible to obtain strong focusing by departing from azimuthal symmetry for the field index n and by going over to an index n which would be a function of the azimuth θ, with a frequency and an amplitude of oscillation of sufficiently large value (fig. 8).

One obtains the dependence $n(\theta)$ by dividing the annular magnet into a number of succesive sectors, in which the magnetic field either rises

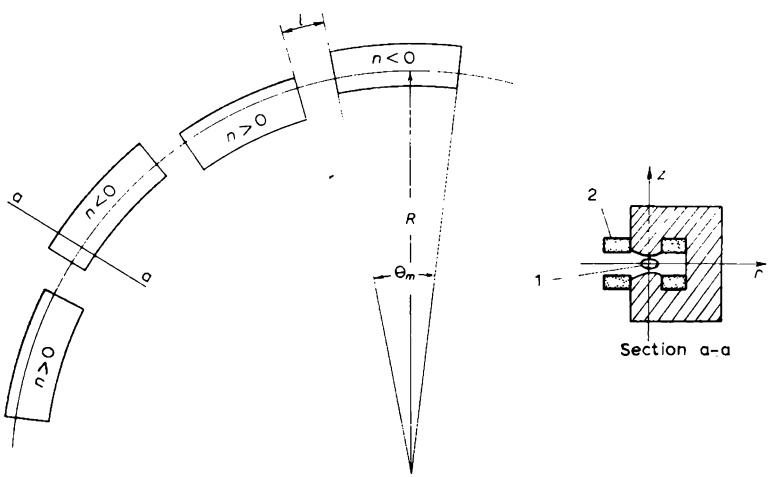

Fig. 9. Schematic example of a strong-focusing synchrophasotron. θ_m-angular dimension of a magnet block, l-length of the field free interval, 1) cross-section of vacuum chamber, 2) magnet coils.

sharply with the radius ($n \ll -1$) or falls sharply with it ($n \gg 1$, see fig. 9).

To understand qualitatively the effect of strong focusing, let us consider for instance the motion along the vertical z. In the sectors with $n \gg 1$, the particles will experience focusing, in the sectors with $n \ll -1$,

defocusing. However, the result may be a strong focusing effect, if one fulfils a few subsidiary conditions. This is due to the fact that, in defocusing sectors, the particles depart from the equilibrium position $z = 0$ and come into the following focusing sectors with a larger mean deviation z. On the contrary, in focusing sectors under the influence of the restoring forces the particle begins to approach the equilibrium position and travels in the following defocusing sectors at a smaller mean distance from $z = 0$. Insofar as the force acting is proportional in the first approximation to the displacement z, that is $F_z \sim m\omega^2 nz$, the focusing will exceed the defocusing, the resulting force being greater, the more n exceeds unity. As for the radial motion, the roles of the focusing and defocusing sectors are exchanged compared with the vertical motion but the overall effect remains the same.

The orbit stability criteria in a strong focusing accelerator, and in general in fields without azimuthal symmetry, are much more complicated than the simple inequalities we have obtained above, (4.6). For this reason, the study of orbit stability in the most recent magnetic systems, some of which have quite an intricate configuration, constitutes

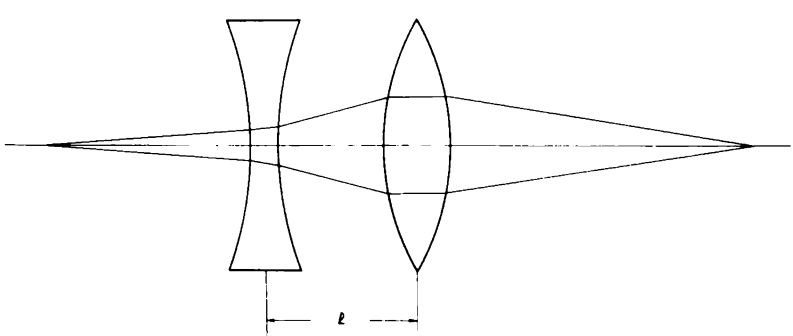

Fig. 10. Optical analogue of strong focusing (schematic).

an important part of the accelerator theory. We will say, as a brief comment, that it is in general necessary in order to get stability in both x and z directions to fulfil the condition $N \approx 2$ to $3\sqrt{|n|}$ where N is the number of periods in one revolution. For instance, when $N \approx 40$–50, the field index $|n|$ must be chosen in the range $|n| \approx 200$–400. With

these conditions, the frequency of betatron oscillations will be 5–10 times the revolution frequency.

The strong focusing effect we have considered represents in itself only a particular case of a general principle of dynamic stability applied to the motion of particles in magnetic systems. An optical version of strong focusing has been known for a long time, for instance; it consists of a system of two lenses, one divergent and one convergent of equal strength which act on a light beam as a convergent lens (fig. 10).

This is easily seen from the usual lens equation

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}, \tag{4.8}$$

where f_1, f_2, F are the focal lengths of each lens and of the system, and L is the distance between the lenses.

If $f_2 = -f_1 = f$, the quantity $F = |f|^2/L$ is positive and such a system provides a convergent beam.

4.2. *Influence of perturbations of the magnetic field on the motion of the particles*

As the real parameters of an accelerator can never correspond exactly to those calculated, it is very important in practice to know what is the influence on the motion of one or the other of the perturbations, and to evaluate on the basis of these results the precision with which the accelerator should be built. In order to give an idea of how serious small differences in some parameters can be, let us examine briefly the influence of a deviation of the field from the azimuthal symmetry:

$$B_z(\theta) = B_0 + b_z(\theta), \tag{4.9}$$

where $B_0 \equiv B_{0z}$ is the ideal azimuthally symmetric field, and the function $b_z(\theta)$ represents a small asymmetry (fig. 11).

In the equation of betatron oscillations (4.7a) there appears a term related to the positive Lorentz force which acts on the particle with a period equal to its period of revolution. The radial oscillations produced by this force will have the form

$$x_0(\theta) = \frac{R}{B_0} \sum_{k=0}^{\infty} \frac{b_k}{k^2 + n - 1} \cos(k\theta + \beta_k), \tag{4.10}$$

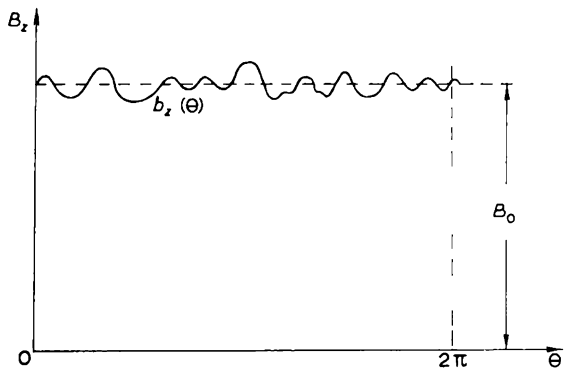

Fig. 11. Azimuthal asymmetry of magnetic field.

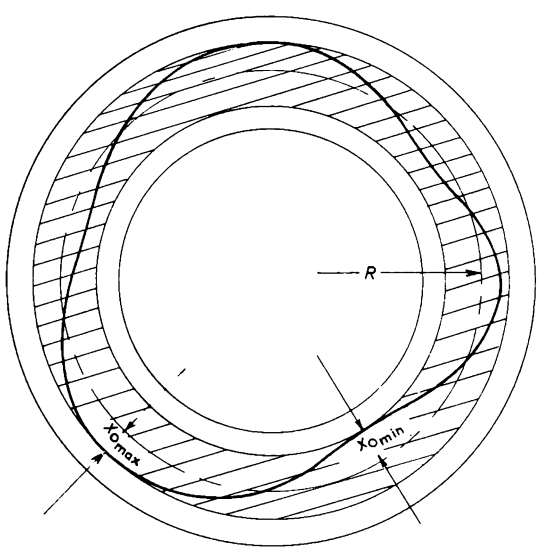

Fig. 12. Typical shape of a perturbed closed orbit. The shaded area corresponds to the part of the aperture which is lost in practice.

where b_k are the coefficients in the expression of $b_z(\theta)$ in a Fourier series, and β_k the phase angles of the various harmonics.

The expression (4.10) represents, clearly, a new closed orbit differing from the equilibrium circle by an amount which increases with the perturbation $b_z(\theta)$. It is evident that because of the azimuthal asymmetry the effective aperture of the chamber is reduced by the amount $|x_{0_{min}}| + x_{0_{max}}$ (fig. 12).

From formula (4.10) we may derive an important conclusion which we may express as follows: the most important deviations of the perturbed orbit are due to the low harmonics of the azimuthal asymmetry. The influence of the remaining harmonics falls rapidly, approximately as the square of their order number. The strong action of the first harmonic can be explained by the fact that this harmonic is the nearest to resonance of the radial betatron oscillations. Strictly speaking, resonance can occur, as we see, only in a uniform field (with $n \approx 0$), where the frequency of radial oscillations coincides with the particle revolution frequency ω, that is with the frequency of the first harmonic of azimuthal asymmetry. In circular accelerators the value of n is chosen around $n \approx 0.6$–0.7 and the resonance we have just considered does not cause immediate danger. Only in accelerators with extended magnets, like the phasotron and especially the cyclotron, where the field is nearly uniform, can the precession of the orbit due to the resonance with the first harmonic of the azimuthal asymmetry be extremely dangerous. It is easy to understand the physical reason for this precession with the help of fig. 14, where it is shown that with any field gradient the orbit is displaced in a direction perpendicular to this gradient.

Let us note that the excursion of the closed orbit in the vertical direction takes place in a quite similar manner. The vertical co-ordinate z_0 of the perturbed orbit will be described, instead of by eq. (10), by the following equation

$$z_0(\theta) = \frac{R}{B_0} \sum_{k=0}^{\infty} \frac{b_k}{k^2 - n} \cos(k\theta + \beta_k), \tag{4.11}$$

where b_k is now the Fourier coefficient of the radial component $b_x(\theta)$ of the perturbing field. The most serious is again the first harmonic, but the resonance appears only if $n = 1$.

These examples, of course, do not exhaust all possible cases of the various resonance phenomena caused by field defects. These questions are specially important in strong focusing accelerators where quite small differences between real and calculated parameters may lead to dangerous resonances. This is why considerations relating to tolerances have a primary importance in the choice of the whole set of parameters of the accelerator. It is the task of the theory to find a compromise between the requirements of sufficiently high efficiency and reliability, on the one hand, and the technical possibilities on the other. At any rate for the normal operation of a strong focusing accelerator it is essential to machine and mount all parts of the magnet with a high degree of accuracy.

4.3. *The acceleration regimes of the particles*

Besides the study of the stability of betatron oscillations in ideal and non-ideal magnetic fields, the analysis of the various ways of increasing particle energy is also a subject of accelerator theory. Perhaps the main question here is the stability of the acceleration regime, that is the realization of conditions under which small deviations of the motion from the ideal will not perturb the operation of the accelerator significantly.

As a rule, in present day accelerators, the 'resonance regime' of acceleration is used in which the increase of particle energy is obtained with the help of a high frequency accelerating voltage $V \cos \omega_0 t$, applied to a gap placed somewhere along the orbit. The resonance condition lies in the fact that the quantity ω_0 is made equal to the frequency of revolution of a particle which we shall call equilibrium (or synchronous).

$$\omega_0 = \omega_s , \tag{4.12}$$

the index s refers here to the equilibrium values (in the more general case $\omega_0 = q\omega_s$, where q is an integer, called the harmonic number of the acceleration). Such a particle accomplishes one turn in the time of one period of the high-frequency voltage, and therefore receives an accelerating impulse at regular intervals. To respect condition (4.12) the influence of the increase in energy of the equilibrium particle must be compensated, either by varying the frequency $\omega_0 = \omega_0(t)$ with time (phasotron), or by changing the magnetic field (synchrotron), or by

changing both simultaneously (synchrophasotron)*. In the course of one revolution the particle energy experiences an increase

$$\delta E = eV \cos \varphi \,, \tag{4.13}$$

where φ is the phase angle at the moment when the particle passes through the accelerating gap (fig. 13).

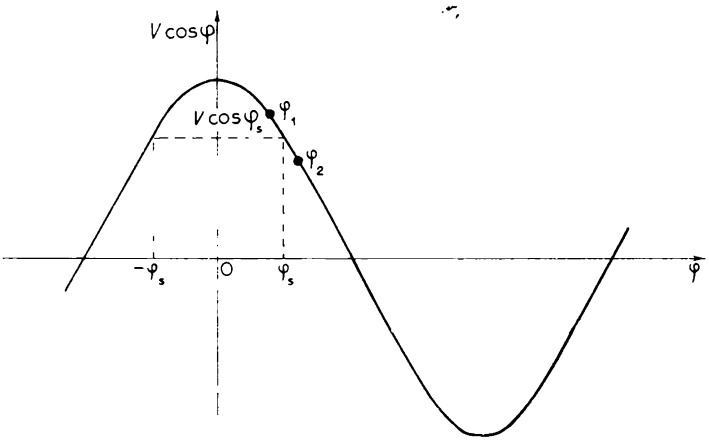

Fig. 13. Explanation of the mechanism of self-focusing.

In order to sustain the resonance regime, one must maintain in the course of time such values for the stable phase angle φ_s, that a particle having this phase at the right energy will always keep it and experience at each revolution the same increase in energy $\delta E_s = eV \cos \varphi_s$.

For the stable phase angle φ_s to be effective, i.e. for the acceleration voltage V to ensure the rate of rise of the energy of the equilibrium particle given by the variation of the radiofrequency $\omega_0(t)$ or the change in the magnetic field $B_z(t)$, some conditions relating these parameters have to be fulfilled. Their choice can however be made with some freedom. For instance, in the case where we maintain ω_0 constant (synchrotron), we will get

* It is also possible to change the harmonic number q: this is the so-called microtron regime (see ch. 7 § 1).

$$\cos \varphi_s = \frac{2\pi c}{\omega_0^2 V \mathscr{K}(E)} \frac{\partial B_z}{\partial t} < 1 , \tag{4.14}$$

where B_z is the field on the equilibrium orbit, and where

$$\mathscr{K}(E) = 1 + \frac{n}{1-n} \frac{1}{\beta^2} ; \quad \beta = \frac{v}{c} . \tag{4.15}$$

Condition (4.14) implies that with a given voltage V, the magnetic field has to change rather slowly, or that with a given rate of rise of B_z the value of V has to be sufficiently large.

The resonance acceleration regime is based on the principle called self-focusing. According to this principle, for relatively weak conditions of the type (4.14), the resonance accelerators have the remarkable property to correct automatically the position of the particle with respect to the stable phase angle of the accelerating voltage. As will be seen from fig. 13, the energy increase at equilibrium can be obtained for two values of the equilibrium phase angle: φ_s or $-\varphi_s$. However only the points φ_s on the falling branch of the voltage sinusoïd give stable acceleration conditions. Let us show this with the help of simple qualitative considerations. Let a particle 1, having the equilibrium energy E_s, pass through the accelerating gap with a phase angle $\varphi_1 < \varphi_s$ that is somewhat smaller than that of the equilibrium (synchronous) particle. Such a particle will receive more energy than is necessary, and the period of its revolution will be bigger than the equilibrium value. As a result, the next time particle 1 will pass through the accelerating gap with a phase that will be nearer to φ_s than the initial phase φ_1. The same way of reasoning can be applied to a particle 2, which goes through the accelerating gap a little later than the equilibrium particle.

Let us consider now a particle going through the gap with the correct phase, but with an energy which will not ensure equilibrium, for instance let us have $E < E_s$. Then its period of revolution will be smaller than for the equilibrium condition, and it will pass through the gap earlier and receive more energy.

In this manner a particle with an incorrect energy value E and a correct phase will return to the equilibrium energy value E_s, just as a particle with an incorrect phase angle φ but a correct energy will return to the stable phase angle φ_s. It is easy to show that, in contrast

to the case we have just considered, excursions from the second resonance phase $-\varphi_s$ will grow, which means that this phase angle is not stable.

Because of this property of self-focusing, or phase-stability, the energy of a non-equilibrium particle performs stable oscillations around the increasing equilibrium energy, and its mean value increases at the same time. These oscillations of phase, energy and also orbit radius are known as synchrotron oscillations, and have a frequency Ω considerably smaller than that of betatron oscillations. For this reason these two types of oscillation can usually be treated independently.

The term synchrotron oscillation may take its origin from the fact that the name synchrotron is sometimes applied quite generally to any resonance cyclic accelerator with self-focusing. The word synchrotron reflects a basic property of such accelerators, the synchronism between the revolution frequency of the particle and the frequency of the accelerating field. The term phase oscillation, which is also found in the literature, seems less appropriate as, strictly speaking, betatron oscillations also are accompanied by some oscillations of the phase. Besides, this term does not reflect the whole range of phenomena associated with the self-focusing regime, such as the oscillations of radius and energy of the particle, which take place together with the phase oscillations.

The characteristic properties of the resonance regime, such as self-focusing and synchrotron oscillation, are preserved, although with some modifications, when one goes over to strong focusing. In fact self-focusing appears because of the dependence of particle revolution frequency on magnetic field and energy $\omega = \omega(E)$. One can write this dependence in the form

$$\frac{\Delta\omega}{\omega} = -\mathscr{K}(E)\frac{\Delta E}{E}, \tag{4.16}$$

where the coefficient $\mathscr{K}(E)$ characterizes the efficiency of the self-focusing action and is equal to the expression given in (4.15) for the azimuthally symmetric field. Equation (4.16) is valid for weak as well as for strong focusing, only the sign and the absolute value of $\mathscr{K}(E)$ for the two types of accelerators may differ.

The study of the self-focusing properties of accelerators of various types in the presence or absence of perturbations forms an important chapter of accelerator theory. The analysis of a few special modes of acceleration can then be derived, e.g. crossing through the so-called critical energy, the capture of the particles in the acceleration regime and some other mechanisms capable of increasing particle energy.

In smaller machines, in addition to and in combination with the resonance method, the induction or betatron regime is also used. In this case an electric rotational field slowly varying with time is applied along the orbit with the help of special cores carrying a magnetic flux variable with time and encircling the orbit. When the radius of the equilibrium betatron orbit R remains constant the energy increase follows the law

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{e^2R^2B_z}{E}\frac{\partial B_z}{\partial t}. \tag{4.17}$$

On the other hand, when $r = R = \text{constant}$, it follows from the laws of electromagnetic induction that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{eRv}{2c}\frac{\partial \bar{B}_z}{\partial t}, \tag{4.18}$$

where $\bar{B}_z$ is the mean field inside the orbit. Combining (4.17) and (4.18) we get the condition which the magnetic field of the betatron must satisfy in order that the radius of the equilibrium orbit should not vary:

$$\frac{\partial \bar{B}_z}{\partial t} = 2\frac{\partial B_z}{\partial t} \tag{4.19}$$

or

$$\bar{B}_z(t) = 2B_z(t) + \text{const.} \tag{4.20}$$

In general the constant appearing in (4.20) is equal to zero, that is, $\bar{B}_z(t)$ and $B_z(t)$ vary with the same phase. Equation (4.19), often called the 2:1 condition, shows that in order to get $R = \text{const.}$, the quantity $\partial \bar{B}_z/\partial t$, which determines the rate of increase of the particle energy must be equal to twice the quantity $\partial B_z/\partial t$, on which depends the variation of the orbit curvature for a given energy. If $(\partial \bar{B}_z/\partial t) < (2\partial B_z/\partial t)$, the orbit will shrink, in the opposite case, it will expand.

4.4. *Particle injection into and extraction from accelerators*

It is possible to divide the motion of the particle in an accelerator in the following three stages:

The first stage (injection) consists of the introduction of the particles and their capture in the acceleration regime.

The second stage is the acceleration period proper (in one of the regimes of resonance or induction acceleration).

The third stage (extraction) is the displacement of the particles on to a target or their ejection out of the machine in order to perform physics experiments.

The construction of accelerators is determined fundamentally by the requirements arising from features of the behaviour of the particle in the second and basic stage, which occupies the major part of the acceleration cycle. However, the first and third stages appear to be extremely important from the point of view of obtaining maximum beam intensity and the optimum conditions for performing the experiments with this beam. Both injection and extraction of the beam have definite peculiarities compared with the 'normal' process of acceleration.

It is necessary to take them into account when designing accelerators and choosing the type of construction and operation. In machines like cyclotrons and phasotrons, the resonance regime of acceleration begins immediately after the ions are produced in the source. In ring-shaped machines, on the contrary, the particles have to be accelerated previously to some energy, depending on the type and scale of the machine. This takes place in a special accelerator called the injector. The latter can consist, in some cases, of a relatively simple system, for instance an electron gun (for betatrons or small synchrotrons). In larger accelerators, however, the injectors are large and intricate devices, as for instance electrostatic van de Graaf generators or linear accelerators (linacs).

During acceleration in linacs, phenomena similar to self-focusing in cyclic accelerators take place. Non-equilibrium particles with defined initial conditions are captured in the acceleration regime and perform oscillations around the position of equilibrium. In contrast with cyclic accelerators, where many phase (synchrotron) oscillations take place during the acceleration period, this time has generally a length of the order of one phase oscillation in linacs.

The characteristic property of linacs seems to be that the fulfilment of the phase stability condition (of the axial motion) is found in contradiction with the stability of the transverse (radial) motion. This fact appears clearly if we transform to a system of co-ordinate moving along with the particle in equilibrium. In this system there will correspond to the electromagnetic accelerating field an electrostatic one in which the equilibrium particle is at rest, whereas the non-equilibrium ones move around. In this case one can apply the well known theorem of Hirnshaw [28], according to which the electrostatic potential in vacuum cannot have maxima or minima, with the exception of 'saddle'-points. This means that if motion of the particle in some direction is stable, motion of the same particle in a perpendicular direction will be unstable.

For this reason one is forced to use special means in order to ensure radial focusing in linacs. One can cover the apertures at the entrance of the tubes with special focusing grids, or apply a supplementary focusing axial magnetic field, or also make use of quadrupole lenses, that is of the strong focusing system.

In order to make the injection and subsequent capture into the acceleration regime sufficiently effective the parameters of the beam have to satisfy certain stringent conditions at the entrance of the accelerator, with respect to transverse dimensions, as well as to angular and energy spread. For the transfer of the beam from the injector to the accelerator special ion- or electron-optical systems are used consisting of bending and focusing magnetic or electric fields which will also give the beam the necessary form.

After the particles have been brought from the injector into the machine with the help of this injection system, they move along for some time in a rather complicated manner, after which some of them are finally trapped into the acceleration mechanism, whereas others are not and get lost on the walls of the vacuum chamber or the injection device.

Looking at the injection question from a general and theoretical point of view, one can guide oneself with the help of the theorem of Poincaré which is well known in mechanics [29]. It states that the trajectory described in phase space by the point representing the system has a quasiperiodic character, in the sense that towards the end of some time interval the system will return to the initial state with any desired degree of accuracy. Applied to injection into a ring-shaped accelerator, it means

that particles introduced into a field not varying with time will necessarily return to the injector or injection device after a certain number of revolutions and get lost in the impact. This is why it is necessary, in order to ensure the trapping of the particles on the orbits, to modify the conditions of motion in some way with time by varying either the magnetic guiding field or the electric injection field.

The scheme known as multi-turn injection is used in betatrons, synchrotrons, weak focusing synchrophasotrons. With this procedure the orbits around which the injected particles perform their oscillations, are displaced away from the injector or injecting device by the growth of the magnetic field with time. This injection lasts for a time corresponding to say some tens of particle revolutions (as for instance in the case seen earlier at the end of § 3), so that the particles get actually stored in the vacuum chamber ready for subsequent acceleration.

Single turn injection is used in strong focusing synchrophasotrons. In this case particles are let in for about the duration of one revolution, after which the voltage on the deflector is removed. When the particles come back to this point after their first revolution, the field has varied and they are trapped in their orbits.

Injection is accompanied by a few other phenomena, as for instance the Coulomb repulsion between particles, scattering against residual gas in the vacuum chamber, etc. These phenomena are most dangerous just at this moment as the particle energy is still small and the transverse beam dimensions are a maximum.

Computation of the injection process and comparison with experiment may well be a complicated task. For instance, the interaction of the particles may be beneficial, as for the capture of the electrons in the betatron regime, since they allow the electrons to avoid the injector (an electron gun), which is necessarily located in the working region of the magnetic field.

A very important aspect in the design and operation of accelerators is the way the accelerated beam is used. Here we may distinguish two possibilities.

The first one is to let the beam strike at the end of acceleration a special target located inside the vacuum chamber. This is done when one wants either to investigate the interaction of the beam with the target material or to use the secondary particles created in the target

for physics experiments. In particular with electron synchrotrons it is not the beam that is used for physics experiments but the gamma rays obtained as 'Bremsstrahlung' when the beam is intercepted with a target.

The second possibility is to extract the accelerated beam from the machine with special devices in order to utilize it to perform various experiments. The effective extraction of an appreciable fraction of the accelerated particles in a well focused beam is a difficult task. It requires a means for perturbing the basic motion of the particles in a short interval of time so that they are compelled to go out of the machine in a given direction.

The following method is applied in large accelerators to achieve this. The beam is made to fall at the end of the acceleration on to a special target, the traversal of which causes only a small energy loss to the beam. As a result the instantaneous orbit around which the betatron oscillations take place shrinks suddenly to a smaller radius. The particles performing these oscillations around the new orbit enter during their first turn into the gap of a special perturbing magnet installed in a field-free section. Here they experience a sharp deflection and come into another bending magnet which finally brings them out. It is necessary, with this method, to avoid bringing the particles into the stray field of the perturbing magnet, as this would lead to scattering and make it difficult to focus them and bring them out in the desired direction. Other ways of extracting also exist based on a modification of the orbit by special magnetic fields. To strengthen the action of these perturbations one can use some resonance phenomena which occur during the motion of the particle in the magnetic field of the accelerator (see ch. 7 § 1.3).

Extraction has some features in common with injection from the point of view of the details of the motion as well as that of the special magnets and other devices necessary to focus the beam and adjust its parameters. The methods of extraction as of injection display a large diversity and depend upon the type of accelerator and its size, the type of particle, etc. The theory of these methods involving elements of electron optics, mass spectrography, etc., are not directly related to the motion of the particles in the accelerator proper and will not in general be considered here.

In principle, it is the object of acceleration theory to study the classical equations of motion in an external magnetic field, composed of the sum of the wanted field and the perturbations. When computing the influence of the latter, which are usually not known 'a priori', it is often impossible to restrict oneself to a pure dynamic approach and one is forced to resort to statistical probability evaluations.

Let us note that quantum effects related to particle motion in accelerators are negligibly small up to the highest conceivable energies. Only under certain circumstances will it be necessary to take into account the quantum statistical character of the radiation emitted by relativistic electrons in magnetic fields (see ch. 5 and 6). We may remark, in this connection, that this radiation not only has a great importance in electron accelerators, but also constitutes in itself an interesting physical phenomenon.

In accordance with the specific periodic character of cyclic accelerators the basis of the theory of betatron oscillations will be the study of differential equations with periodic coefficients. These equations have the following form after some simplifying assumptions

$$\frac{d^2x}{d\theta^2} + g(\theta)x = \sum_{\substack{m+n>1 \\ m,n\geq 0}} a_{mn}(\theta)x^m \left(\frac{dx}{d\theta}\right)^n, \tag{4.21}$$

where $g(\theta)$, $a_{mn}(\theta)$ are functions of the azimuth with the same periodicity as the magnetic system.

The point of departure for the theory is a set of linear equations belonging to the class known as Hill equations, for which a general theory and well developed methods of computation exist.

The main task of the theory will be to study particle stability to determine the frequencies of the associated oscillations, to find the permissible ranges for the values of the parameters and the tolerances of various perturbations. In the same sense of 'accelerator design' problems, several modifications of perturbation theory have been found useful, such as expansions as a function of a small parameter, average methods and similar ones, making it possible to find and evaluate the influence of dangerous resonance perturbations.

In recent years great attention has been given to the study of nonlinear phenomena which are very difficult to compute because there

does not exist a sufficiently general theory of non-linear equations with periodic (or even constant) coefficients. The non-linear terms on the right-hand side of eq. (4.2) have two basic effects: dependence of the oscillation frequency on the amplitude, and the appearance of supplementary 'non-linear resonances'. These effects, and of course also the linear ones, become more complicated still because of the variation of the parameters with time, the coupling between oscillations with different degrees of freedom, and various other perturbations. Several problems of practical importance, in particular involving dynamic non-linear effects (see ch. 3 § 8), cannot be considered as fully solved yet.

Many theoretical questions have to be solved for the different acceleration regimes. The principle characteristic features of resonance acceleration (self-focusing) are described as a rule by non-linear equations analogous to those of a physical pendulum with an external couple. However, a series of questions, concerning in particular acceleration in strong focusing machines, cannot be studied in the frame of this equation and require the use of much more intricate mathematical tools (critical energy, regulation of the parameters of the high-frequency system by the beam itself, etc.).

An important part of the theory is devoted to the computation of particle losses caused by the action of various perturbing factors, or occurring during transient processes such as capture into the acceleration regime. Strictly speaking this computation should be made on the basis of the equations of motion for the distribution function of the particles for various boundary conditions (determined by the vacuum chamber walls or other external factors).

Generally, it is possible to obtain all the basic information relevant to particle motion in accelerators by considering the motion of a single particle in the given fields without taking into account its interaction with the other particles of the beam. However such an approach can be insufficient in certain circumstances and it may be necessary to compute the interaction of the particles in the beam (e.g. at injection, or for storage devices if the circulating beam is very intense). In this case one must apply the equations of motion taking into account the action of the electromagnetic field in the co-ordinate system of the particles.

In recent years some general considerations of statistical mechanics, related to the properties of phase space (in particular Liouville's

theorem), have found a new and quite fruitful application in accelerator theory. Let us consider a group of k particles moving under the action of forces which can be described by a Hamiltonian $\mathscr{H}(q_1, \ldots q_k, p_1, \ldots p_k, t)$ where q_1, p_1 are canonical variables of position and momentum. Then Liouville's theorem states that the points representing the state of the system which move about in a $2k$-dimensional phase-space behave as an incompressible fluid, that is, the density of these points does not change during the motion of any determined volume in this space, although the shape of this volume can vary substantially.

It is important to note that any system with external electric and magnetic fields acting on charged particles is Hamiltonian, if one neglects radiation effects. Systems of interacting particles, which are characterized by their accompanying fields and their trajectories, are usually Hamiltonian also.

The concept of phase-space finds one of its principal applications in the computation of the intensity of the beam entering the accelerator from the injector side. The injection device must be designed and built in such a way that the phase-space volume occupied by the beam leaving the injector (emittance) should be as nearly equal as possible to the part of this phase-space corresponding to the particles accepted by the accelerator proper (acceptance or admittance). Analogous problems appear when considering 'cascade' systems where the beam leaves one accelerator to enter another, or storage devices which are fed with maximum beam current from a special accelerator.

One of the most interesting uses of Hamilton's formalism was made in the study of the radio frequency acceleration regimes. When problems already solved concerning injection were considered in this manner statistical mechanics methods made it possible to understand a number of phenomena in a new way and to propose modifications to the known acceleration systems. Thus the concept of phase-space has indicated a method of acceleration based on what is called the 'phase-displacement' of the particles (see ch. 7 § 4). Another example is the application of Liouville's theorem to the storage of large currents in systems with a constant magnetic field, examined in ch. 7.

CHAPTER 2

BETATRON OSCILLATIONS OF PARTICLES IN IDEAL PERIODIC MAGNETIC FIELDS

The class of magnetic fields used to guide the motion of particles in present-day cyclic accelerators is rather wide, and the particle trajectories are usually complicated. However, machines of the various types have some features in common which allow an approach to their theory from a more or less unified point of view.

To begin with, one such property is that the electrical field is generally small in comparison with the magnetic guide field in accelerators, so that the particle energy does not vary noticeably in the course of one revolution. This is why it is possible, in present-day accelerators, to separate and study independently three types of processes. In the first stage of theory it is essential to examine the motion of a particle with constant energy in a magnetic field constant in time, and to consider effects depending on the electric field as slow adiabatic changes of the parameters of the motion. After this one studies these slow changes brought about by the acceleration process which usually manifest themselves in synchrotron oscillations due to the self-focusing mechanism (see ch. 1 § 4 and ch. 4). Finally the third process concerns the increase in energy 'in the average', which corresponds to the variation of the magnetic field or the frequency of the accelerating voltage with time.

It is important to stress that this division is based on the large difference of duration of these processes. As an example the period of synchrotron oscillations in the synchrophasotron of the J.I.N.S. is 100 to 1000 times larger than the period of revolution of a particle; in the whole acceleration cycle there are only a few thousand synchrotron oscillations.

The guiding fields and the initial conditions which bring about a motion as nearly periodic as possible are the most interesting for the theory. It is this property, which results in a natural manner from the

geometry and the principle of operation of cyclic accelerators, that allows us to consider the motion in a magnetic field not varying with time as the superposition of a periodic part and a small non-periodic one.

Another important property of magnet systems for accelerators is their symmetry. The magnet systems utilized now have usually a plane symmetry, often called median plane, in the vicinity of which lie the trajectories, a feature which presents definite advantages for their construction.

Further, modern magnetic fields have usually an axis of symmetry of some order. The simplest and historically the first field used for an accelerator was azimuthal of the first order. Later, however, new types of magnetic systems were developed consisting of identical structural units (see ch. 1 § 4 and figs. 1 and 9), in the limits of which the field may or may not be azimuthal of the first order. The number N of such elements of periodicity varies in a wide range (from one to a few tens).

In this chapter we shall examine the general properties of the motion of a particle with a constant energy in ideal fields. The perturbations of the motion caused by the inevitable differences of the properties of real systems compared with the calculated ones will be the subject of ch. 3.

1. CLOSED ORBITS AND NATURAL SYSTEM OF CO-ORDINATES

As the particle motion is nearly periodic, a basic concept in acceleration theory will be the orbits of the particles (often called also closed, instantaneous, periodic or equilibrium orbits). By this we mean a particle trajectory with a given energy in a given magnetic field which will close on itself. The existence of such a trajectory is not in itself evident and in any case imposes some conditions on the geometry of the magnetic field.

The following general statement is however valid [1]: if, in a magnetic field not varying with time, one of several arbitrary closed curves of a given length covers the experimental beam, then this curve is an orbit for particles of a certain energy.

To show this let us consider a closed curve C and write the integral of action along this curve

$$I = \oint_C \left[-m_0 c^2 \sqrt{1-\beta^2}\, \mathrm{d}t + \frac{e}{c} A\, \mathrm{d}r \right], \tag{1.1}$$

where A is the vector potential of the magnetic field B. The condition for the curve C to be an orbit is evidently that the variation δI tends towards zero, i.e. that the equation of motion be fulfilled in each point of the curve. The first term under the integral is always constant because of the conservation of energy, so that the integral does not depend on the choice of curve C, due to the equality in length of the considered curves. There remains

$$\delta I = \frac{e}{c} \delta \oint_C A \, dr = 0 . \tag{1.2}$$

Now, because of Stoke's theorem

$$\oint_C A \, dr = \int_S \text{rot } A \, ds = \int_S B \, ds = \Phi , \tag{1.3}$$

where S is the surface surrounded by curve C and Φ is the scalar magnetic flux through this surface. So the necessary and sufficient condition for the closed curve C to be an orbit with length Π, is

$$\delta \Phi|_{\Pi = \text{const.}} = 0 . \tag{1.4}$$

One can give examples where condition (1.4) cannot be fulfilled, e.g. if the magnetic flux attains only its greatest value, but not an extremum, when curve C is made to vary. In such a case none of the curves considered is an orbit. A situation of this type is shown in fig. 14, where the magnetic field is everywhere perpendicular to the plane of the figure

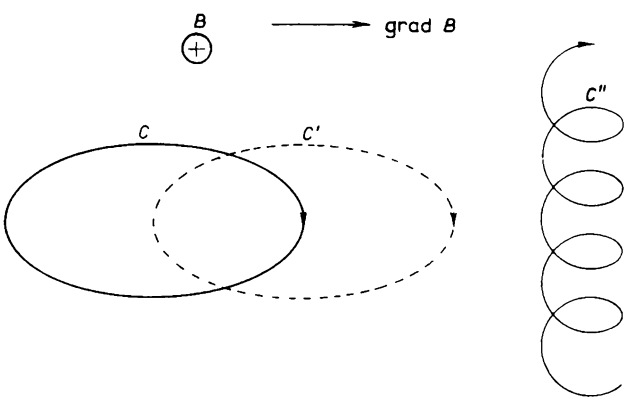

Fig. 14.

and grows in absolute value in the direction of the arrow (only plane orbits are drawn). The closed curve C may have any form whatsoever - we shall always increase the magnetic flux it embraces, by moving it to the right into position C'. In this example, the flux has no extremum and there is no closed orbit (in this kind of field the particle experiences the so-called gradient-drift whereby it moves along a trajectory of the type C'').

In many cases one succeeds in determining the orbits by simple physical considerations. So in the case of azimuthal symmetry, the orbits are simply circles of radius $R = pc/e|B|$, where p is the absolute value of the total particle momentum, and B the component of the magnetic field perpendicular to the orbit. In synchrophasotrons one can generally consider the orbits as being composed of arcs of a circle and straight lines.

In more intricate fields the discovery of even the ideal orbit is by no means so trivial and presents definite mathematical difficulties, e.g. in the case of ring-phasotrons. We shall come back to this question in ch. 7 § 3. For the moment we shall assume that the orbit exists and all its characteristics are known.

At a first glance it may appear that the use of the closed-orbit concept is somewhat artificial, as a particle, even with specially chosen initial conditions, cannot move along it for a long time because of various

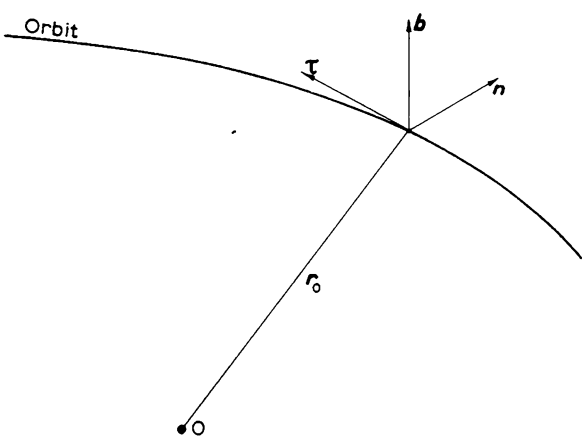

Fig. 15. The co-ordinate system moving with the particle.

perturbing factors (gas scattering in the chamber, action of the accelerating electrical field, etc.). However, representation of the motion as the superposition of a periodic part (motion along the orbit) and small deviations from it really has a physical sense and allows a clear interpretation of the complicated processes occurring in accelerators. Finally,

in the real machine, the extremely important condition of the stability of the closed orbit should be fulfilled. By this we mean the requirement that sufficiently small perturbations lead to only small displacements of the actual trajectory from the closed orbit.

Before going on to the study of this condition which forms the basis of this chapter, let us examine some general properties of closed orbits which will determine the results obtained later in many cases.

It follows from the symmetry properties of the magnetic system that the orbit is a periodic curve in space, i.e. that all its characteristics are periodic functions of the length of arc σ with a period Π/N, where Π is the orbit length, and N the number of periods. To these characteristics belong, in the first place, the curvature $K(\sigma)$ and the torsion $\kappa(\sigma)$, which is evidently zero for a plane orbit ($\kappa \equiv 0$). In arbitrary periodic systems it is convenient to study small deviations from the closed orbit in the so-called natural system of co-ordinates constituted by the unit vectors of the normal $\boldsymbol{n}(\sigma)$, tangent $\boldsymbol{\tau}(\sigma)$ and binormal $\boldsymbol{b}(\sigma)$ to the orbit as reference (see fig. 15). This system is triorthogonal and right-handed*.

If the vector equation of the orbit has the form $\boldsymbol{r} = \boldsymbol{r}_0(\vartheta)$, these vectors and their derivatives are bound by the well-known relations of Serret–Frenet [2]

$$\begin{aligned} \boldsymbol{\tau}(\vartheta) &= K_0 \frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}\vartheta}\,; & \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\vartheta} &= -\frac{K(\vartheta)}{K_0}\boldsymbol{n}(\vartheta)\,; \\ \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}\vartheta} &= \frac{1}{K_0}(K\boldsymbol{\tau} - \kappa\boldsymbol{b})\,; & \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}\vartheta} &= \frac{\kappa(\vartheta)}{K_0}\boldsymbol{n}(\vartheta)\,, \end{aligned} \tag{1.5}$$

where $K_0 = 2\pi/\Pi$, and where we have taken as independent variable the 'generalized' azimuth

$$\vartheta = \frac{\sigma}{\Pi} \cdot 2\pi = K_0\sigma\,, \tag{1.6}$$

which coincides with the usual azimuthal angle for plane orbits.

The equation of motion of the particle in a magnetic field $\boldsymbol{B}(\boldsymbol{r})$ may be written in the vector form

* The order of the unit vectors is chosen in such a way that in the case of azimuthal symmetry the natural co-ordinates become cylindrical co-ordinates.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)=\frac{e}{c}\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\boldsymbol{B}(\boldsymbol{r})\right], \tag{1.7}$$

where e is the particle charge, and m its total (relativistic) mass. If the velocity remains constant then the quantity $\mathrm{d}\vartheta/\mathrm{d}t = K_0|\mathrm{d}\boldsymbol{r}_0/\mathrm{d}t| = \text{const.}$ for a particle moving on the orbit, and we obtain from (1.7)

$$K\boldsymbol{n} = -\frac{e}{pc}[\boldsymbol{\tau}\boldsymbol{B}(\boldsymbol{r})]. \tag{1.8}$$

This will give us simple expressions for the components of the field along the orbit:

$$B_x \equiv 0\ ; \qquad K \equiv -\frac{e}{pc}B_z\,, \tag{1.9}$$

where $p = m\left|\dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|$ is the particle momentum, and where

$$B_x = \boldsymbol{B}\boldsymbol{n}\ ; \qquad B_z = \boldsymbol{B}\boldsymbol{b}\,. \tag{1.9a}$$

For a circular orbit of radius $R = K^{-1}$, the second expression takes the well-known form*

$$pc = -eB_zR\,. \tag{1.10}$$

If the orbit is not a circle, the particle momentum is linked with the mean value of the B_z component on the orbit in an analogous manner. In fact, integrating (1.9) and keeping in mind that $\oint K\,\mathrm{d}\sigma = 2\pi$ we get**

$$cp = -\frac{e\Pi}{2\pi}\langle B_z\rangle \equiv -\frac{e}{K_0}\langle B_z\rangle\,, \tag{1.11}$$

in which the sign $\langle\ \rangle$ indicates the mean value along the orbit

$$\langle B_z\rangle = \frac{1}{\Pi}\oint B_z\,\mathrm{d}\sigma\,. \tag{1.12}$$

* The minus sign in formulae (1.9) and (1.10) reflects the fact that a positively charged particle moves anti-clockwise if one looks in the direction of the field.

** We will not consider orbits with crossovers, where $\oint K\,\mathrm{d}\sigma = 2\pi k, k = 1, 2, 3, \ldots$

It is easy to obtain from this an expression for the period of revolution of the particle along the orbit

$$T = \frac{2\pi E}{e\,|\langle B_z\rangle|\,c}\,; \qquad E = mc^2\,. \tag{1.13}$$

2. EQUATIONS FOR SMALL DEVIATIONS FROM THE CLOSED ORBIT

As follows from ch. 1, it is particularly important in accelerator theory to study trajectories close to the orbit, because the stability of small deviations is the principal condition for successful operation of the machine. As regards the question of motion stability 'as a whole', it is, at least at present, more of analytical than of practical interest.

In this connection we shall study the stability by the method of Liapunov, i.e. search for stability criteria for the solutions of the linear differential equations which describe small displacements from the orbit. As the orbit characteristics are periodic azimuth functions, it is natural to expect that the linearized equations of motion will have periodic coefficients. The theory of the stability of such equations is well developed [4, 5] and makes it possible to look at this problem from a sufficiently general standpoint.

The most important case in practice is that of plane orbits ($\kappa \equiv 0$). We shall consider just this case in detail in order to simplify the intermediate formulae, and because it applies to the overwhelming majority of recent machines. We shall derive the equations of motion under much more general assumptions in appendix A. To write down the equations of motion in the natural co-ordinate system [6, 7] let us first note that the radius vector of an arbitrary point, lying less than a radius of curvature away from the orbit, may be described in the form

$$\boldsymbol{r} = \boldsymbol{r}_0(\vartheta) + x\boldsymbol{n}(\vartheta) + z\boldsymbol{b}(\vartheta)\,. \tag{2.1}$$

We shall consider the quantities $u^i = x, \vartheta, z$ as the curvilinear co-ordinates of this point. As we know, to the co-ordinates u^i (see for instance [8]) correspond a set of, in general, non-orthogonal vectors $\boldsymbol{a}_i$, determined in the present case by the equations*

* Let us state, to avoid misunderstandings, that the vectors $\boldsymbol{n}, \boldsymbol{\tau}, \boldsymbol{b}$ do not constitute a true system of co-ordinates, i.e. an infinitely small increase $d\boldsymbol{r}$ of vector $\boldsymbol{r}$ cannot be expressed in the form $\boldsymbol{n}\,dx + \boldsymbol{\tau}\,d\vartheta/K_0 + \boldsymbol{b}\,dz$.

$$\boldsymbol{a}_1 = \frac{\partial \boldsymbol{r}}{\partial x} = \boldsymbol{n}\ ; \qquad \boldsymbol{a}_2 = \frac{\partial \boldsymbol{r}}{\partial \vartheta} = \frac{1+Kx}{K_0}\,\boldsymbol{\tau}\ ; \qquad \boldsymbol{a}_3 = \frac{\partial \boldsymbol{r}}{\partial z} = \boldsymbol{b}\ . \quad (2.2)$$

The components of the metric tensor $g_{ik} = \boldsymbol{a}_i \boldsymbol{a}_k$ have in this system the following form

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1+Kx}{K_0}\right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.3)$$

and the determinant $g = \det g_{ik}$, defining the volume element $\mathrm{d}V = g^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}\vartheta$ is equal to

$$g = \left(\frac{1+Kx}{K_0}\right)^2 . \quad (2.4)$$

It is natural to represent the magnetic field in the form

$$\boldsymbol{B} = B_\vartheta \boldsymbol{\tau} + B_x \boldsymbol{n} + B_z \boldsymbol{b}\ . \quad (2.5)$$

However, to write Maxwell's equations for the components B_ϑ, B_x, B_z one must know how they depend on the covariant and contravariant components of the magnetic field in the chosen curvilinear system. Quite generally, we infer from (2.3) the formulae for the contravariant components

$$B^1 = B_x\ ; \qquad B^2 = \frac{K_0}{1+Kx} B_\vartheta\ ; \qquad B^3 = B_z\ , \quad (2.6a)$$

and for the covariant ones

$$B_1 = B_x\ ; \qquad B_2 = \frac{1+Kx}{K_0} B_\vartheta\ ; \qquad B_3 = B_z\ . \quad (2.6b)$$

As the equations of Maxwell for the field $\boldsymbol{B}$ in the static case have the form

$$\operatorname{div} \boldsymbol{B} = \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{\partial}{\partial u^i} (B^i \sqrt{g}) = 0 \quad (2.7a)$$

and $$\text{rot } \boldsymbol{B} = \frac{1}{\sqrt{g}} \left\{ \left(\frac{\partial B_3}{\partial u^2} - \frac{\partial B_2}{\partial u^3} \right) \boldsymbol{a}_1 + \left(\frac{\partial B_1}{\partial u^3} - \frac{\partial B_3}{\partial u^1} \right) \boldsymbol{a}_2 + \right.$$

$$\left. + \left(\frac{\partial B_2}{\partial u^1} - \frac{\partial B_1}{\partial u^2} \right) \boldsymbol{a}_3 \right\} = 0 , \quad (2.7b)$$

the quantities B_ϑ, B_x and B_z are linked by the relations

$$K_0 \frac{\partial B_\vartheta}{\partial \vartheta} + \frac{\partial}{\partial x} B_x(1+Kx) + (1+Kx) \frac{\partial B_z}{\partial z} = 0 ; \quad (2.8a)$$

$$\frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z} ; \qquad K_0 \frac{\partial B_z}{\partial \vartheta} = (1+Kx) \frac{\partial B_\vartheta}{\partial z} ;$$

$$K_0 \frac{\partial B_x}{\partial \vartheta} = \frac{\partial}{\partial x} (1+Kx) B_\vartheta . \quad (2.8b)$$

In order to relate to the time t the generalized azimuth ϑ which we have taken as independent variable, let us differentiate (2.1) with respect to ϑ (primes will always indicate differentiation with respect to ϑ).

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}\vartheta} = \frac{1}{K_0} \left[\boldsymbol{\tau}(1+Kx) + \boldsymbol{n} K_0 x' + \boldsymbol{b} K_0 z' \right] , \quad (2.9)$$

from which

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \frac{K_0 pc^2}{E} \left[(1+Kx)^2 + (K_0 x')^2 + (K_0 z')^2 \right]^{-\frac{1}{2}} \quad (2.10)$$

where E is the particle energy. Going over to the equations of motion with ϑ as independent variable and projecting them on $\boldsymbol{n}$ and $\boldsymbol{b}$ we get

$$\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}t} x' \right)' = \frac{1+Kx}{K_0} \left(\frac{ecB_z}{E} + \frac{K}{K_0} \frac{\mathrm{d}\vartheta}{\mathrm{d}t} \right) - z' \frac{ecB_\vartheta}{E} ; \quad (2.11a)$$

$$\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}t} z' \right)' = \frac{ecB_\vartheta}{E} x' - \frac{ecB_x}{E} \frac{(1+Kx)}{K_0} . \quad (2.11b)$$

These equations, together with (2.10) and with the expansion of the field components in series of x and z give us the means of studying the

particle motion of any order in x, x', z, z' around an arbitrary curve in space. However in the framework of this chapter it will be sufficient to consider the linearized equations for deviations from the orbit. We introduce a quantity

$$n = -\left(\frac{1}{KB_z}\frac{\partial B_z}{\partial x}\right)_{r=r_0} = -\left(\frac{1}{KB_z}\frac{\partial B_x}{\partial z}\right)_{r=r_0}, \tag{2.12}$$

which contains the derivative of the field in the direction of the normal to the orbit and is a generalization of the usual field index (see ch. 1, eq. 4.3). Now we take into account the mirror symmetry of the field and get, with eqs. (2.8), (2.10) and (1.9)

$$x'' + g_x x = 0 \; ; \qquad g_x = \frac{K^2}{K_0^2}(1-n) \; ; \tag{2.13a}$$

$$z'' + g_z z = 0 \; ; \qquad g_z = \frac{K^2}{K_0^2} n \tag{2.13b}$$

(One admits that there is no longitudinal field along the orbit*). It is important to remark that in this approximation the equations for x and z are fully decoupled, so that the motion appears as a superposition of two kinds of oscillations which we will call betatron oscillations as in the accelerator with azimuthal symmetry.

From (2.13) we see that the restoring force coefficients g_x and g_z are related through the quantity n in such a way that if one is increased at some point the other one is automatically reduced. As already pointed out in ch. 1, this property sets a limit to the focusing which may be attained with an azimuthally symmetric field. However, in a general manner the equations (2.13) have the feature that the restoring force coefficients g_x and g_z depend on the azimuth and this in a manner which can be chosen arbitrarily; in particular the quantity $|n|$ may have very high values (strong focusing). Although the conclusion about a contradiction between the conditions of radial and vertical focusing is in

* Let us state that under these circumstances B_z turns out to be an even function and B_x, B_ϑ odd functions of the z-co-ordinate. The case where one has a longitudinal component with a plane orbit is only possible in an azimuthally symmetrical field ($K = -eB_z/pc =$ const.) as follows from (2.8b).

general valid, it is possible under determined circumstances to attain simultaneous stability of the x- and z-oscillations because the restoring forces, which bring back the particles to the orbit, are much higher than with azimuthal symmetry.

One can come to the same conclusions in the much more general case of three-dimensional orbits, i.e. for fields without mirror symmetry. The corresponding linearized equations have then the form (see appendix A):

$$x''+g_x x = \left[\frac{e}{pcK_0^2}\frac{\partial B_z}{\partial z} - \frac{\kappa'}{K_0}\right]_{r_0} z - \left[\frac{eB_\vartheta}{pc} + 2\kappa\right]_{r_0}\frac{z'}{K_0}\,; \quad (2.14a)$$

$$z''+g_z z = \left[\frac{\kappa'}{K_0} - \frac{e}{pcK_0^2}\frac{\partial B_x}{\partial x}\right]_{r_0} x + \left[\frac{eB_\vartheta}{pc} + 2\kappa\right]_{r_0}\frac{x'}{K_0}\,. \quad (2.14b)$$

where

$$\begin{aligned} g_x &= \frac{K^2}{K_0^2}(1-n) - \frac{\kappa^2}{K_0^2} - \frac{eB_\vartheta\kappa}{pcK_0^2}\Big|_{r=r_0}, \\ g_z &= \frac{K^2}{K_0^2}n - \frac{\kappa^2}{K_0^2} - \frac{eB_\vartheta\kappa}{pcK_0^2}\Big|_{r=r_0}. \end{aligned} \quad (2.15)$$

We see that with an arbitrary torsion and a longitudinal field on the orbit the equations for the x- and z-oscillations are no longer independent. However, in this case also the motion presents some general features which will be examined in § 3.

3. GENERAL STABILITY CONDITIONS FOR THE SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

The solution of system (2.14) is completely determined by four initial conditions and may be given in the form

$$\boldsymbol{X}(\vartheta) = \mathbf{M}(0|\vartheta)\boldsymbol{X}(0)\,, \quad (3.1)$$

where $\boldsymbol{X}(\vartheta)$ is a vector with four components

$$\boldsymbol{X}(\vartheta) = \begin{pmatrix} x(\vartheta) \\ x'(\vartheta) \\ z(\vartheta) \\ z'(\vartheta) \end{pmatrix}, \quad (3.2)$$

and $\mathbf{M}(0|\vartheta)$ a square matrix with 4 rows and 4 columns, the elements of which depend on the form of the magnetic field.

Let us designate by $\mathbf{M}$ simply the matrix pertaining to one period.

$$\mathbf{M} = \mathbf{M}(0|\vartheta_0) \; ; \qquad \vartheta_0 = 2\pi/N \, . \tag{3.3}$$

Then, when the particle has gone through k periods, the vector $X(\vartheta)$ will have the form

$$X(k\vartheta_0) = \mathbf{M}^k X(0) \, . \tag{3.4}$$

Let us express $X(0)$ as a function of the eigenvectors V_i of matrix $\mathbf{M}$ with the eigenvalues λ_i*

$$X(0) = \sum_{i=1}^{4} x_i V_i \, . \tag{3.5}$$

Considering that the eigenvalues of the matrix $\mathbf{M}^k$ are equal to λ_i^k we get

$$X(k\vartheta_0) = \sum_{i=1}^{4} x_i \lambda_i^k V_i \, . \tag{3.6}$$

From this it follows that the necessary condition for stability, i.e. that $|X|$ remains limited when k grows, is given by the inequality

$$|\lambda_i| \leqslant 1 \tag{3.7}$$

(the eigenvalues λ_i may be complex, in general).

Let us now dwell on a very important property of system (2.14), i.e. that it describes a system without dissipation. In fact one can write it in the form

$$X'(\vartheta) = \mathbf{P} X(\vartheta) \, , \tag{3.8}$$

where $\mathbf{P}$ is a matrix built up with the coefficients (2.14). As (2.14a) and (2.14b) do not contain terms in x' and z' respectively, the sum of the diagonal elements in matrix $\mathbf{P}$, indicated mathematically by Sp (spur or trace), is equal to zero.

$$\mathrm{Sp}\, \mathbf{P} \equiv 0 \, . \tag{3.9}$$

* The system of eigenvectors V_i is the non-trivial solution of the vector equation

$$\mathbf{M} V = \lambda V . \tag{3.5a}$$

The values $\lambda = \lambda_i$ for which such a solution exists are called the eigenvalues (see for instance [9]).

From the general theory of linear differential equations, it is known [9] that the Wronskian of the system, that is the determinant built up with the four linearly independent solutions $\boldsymbol{X}_{1,2,3,4}$, varies in accordance with the Jacobi formula

$$D(\vartheta) = \det |\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4| = \text{const.} \times \exp(-\int \mathrm{Sp}\,\mathbf{P}\,\mathrm{d}\vartheta) = \text{const.} \quad (3.10)$$

This shows the absence of dissipation in our case. As each of the vectors $\boldsymbol{X}_i$ may be represented in the form (3.1), one concludes from formula (3.10) that:

$$\det \mathbf{M}(0|\vartheta) = 1$$

and in particular

$$\det \mathbf{M} = 1\,. \quad (3.11)$$

The characteristic equation, i.e. the equation for determining the eigenvalues, has the following form, in accordance with (3.5a)

$$\det(\mathbf{M} - \lambda \mathbf{1}) = 0\,, \quad (3.12)$$

where $\mathbf{1}$ is the unit matrix, and appears to be an algebraic equation, in our case of the 4th order. Its coefficients are real inasmuch as the elements of matrix $\mathbf{M}$ are real. For this reason there follows from (3.12) and (3.11)

(a) the transfer matrix $\mathbf{M}$ has in general 4 different eigenvalues,
(b) the quantities λ_i are complex conjugate pairs,
(c) the product of the eigenvalues is equal to

$$\lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1\,. \quad (3.13)$$

From inequality (3.7) with (3.13) we arrive at an important conclusion. The necessary condition for the stability of the solutions of system (2.14) is that the moduli of all eigenvalues of the transfer matrix are equal to one.

$$|\lambda_i| = 1\,. \quad (3.14)$$

This condition is also sufficient, provided that none of the eigenvalues is equal to unity.

Thus to achieve stability the eigenvalues have to be on the unit circle

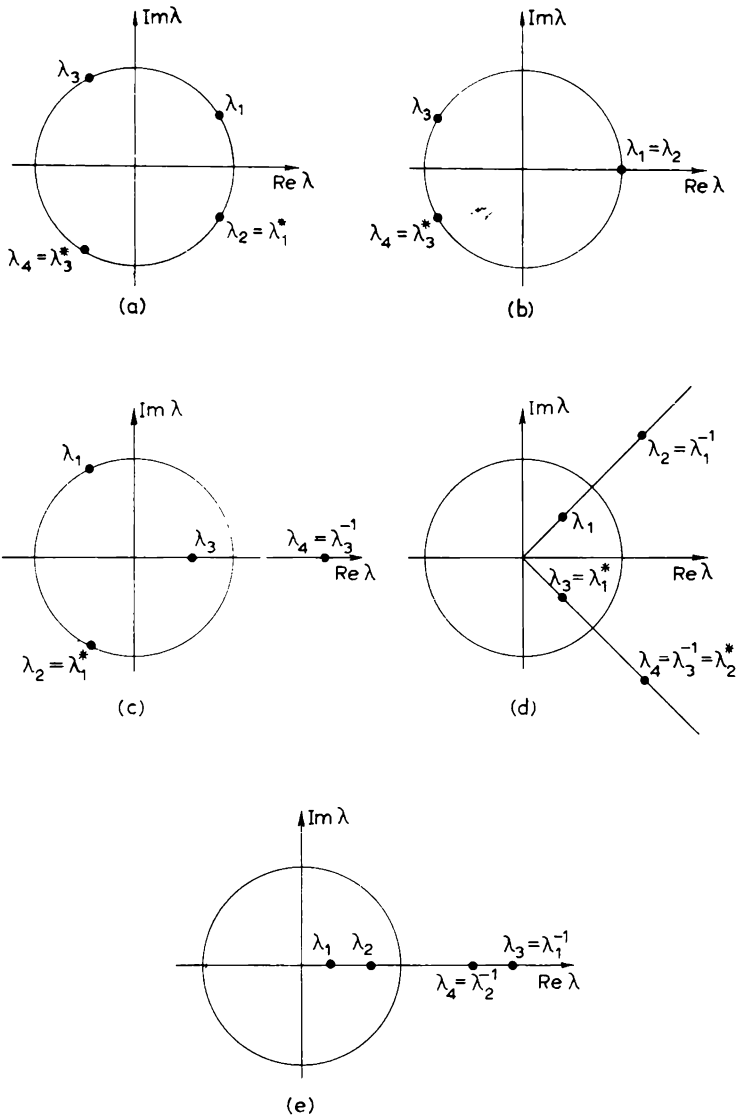

Fig. 16. Schematic location of the roots of the characteristic equation in the complex plane.

in the complex plane. It is then natural to introduce a new quantity μ_i to characterize stability

$$\lambda_i = e^{i\mu_i} . \tag{3.15}$$

As we shall see below, this quantity has a clear physical meaning.

In the complex plane the eigenvalues of the matrix may be located in the following ways (fig. 16):

(a) All four eigenvalues lie on the unit circle and form two pairs of complex conjugate numbers. No one of the quantities λ_i is equal to 1, all quantities μ_i are real.

(b) This case is the same as above, but at least two eigenvalues are identical to 1.

(c) Two eigenvalues lie on the unit circle and are complex conjugates, two others do not lie on the circle and are reciprocal to each other ($\lambda_3 \lambda_4 = 1$).

(d, e) None of the quantities λ_i lies on the circle, the eigenvalues form a complex conjugate pair and a reciprocal pair*.

According to (3.14) the motion will be stable only in case (a) and under certain conditions in case (b).

Let us stress that the eigenvalues do not depend on the azimuth which was chosen as the origin of the period, and are determined by the structure of the magnet system only. In fact, let $\mathbf{M} \equiv \mathbf{M}(0|\vartheta_0)$ and $\mathbf{M}_1 \equiv \mathbf{M}_1(\vartheta_1|\vartheta_1+\vartheta_0)$ be the transfer matrices for two different origins $\vartheta = 0$ and $\vartheta = \vartheta_1$. It is easy to see that

$$\mathbf{M}_1 = \mathbf{M}^{-1}(0|\vartheta_1)\mathbf{M}\mathbf{M}(0|\vartheta_1) . \tag{3.16}$$

If this is introduced into the characteristic equation

$$\det(\mathbf{M}_1 - \lambda \mathbf{1}) = 0 , \tag{3.17}$$

the latter will take the form

$$\det(\mathbf{M}^{-1}(0|\vartheta_1)(\mathbf{M} - \lambda \mathbf{1})\mathbf{M}(0|\vartheta_1)) \equiv \det(\mathbf{M} - \lambda \mathbf{1}) = 0 , \tag{3.18}$$

which is identical to eq. (3.12) for the eigenvalues of matrix $\mathbf{M}$. So both matrices $\mathbf{M}$ and $\mathbf{M}_1$ have the same system of eigenvalues. This is especial-

* From (3.13) it does not follow in general that eigenvalues appear in reciprocal pairs. This property is a consequence of the general theorem of Liapunov-Poincaré, which will be seen further in § 5 and in appendix B.

ly evident for the one-dimensional motion, where the eigenvalues are determined only by the determinant and the trace of the transfer matrix (see eq. 3.20), which do not depend on the origin of ϑ.

We shall consider now in more detail the motion in the case where the orbit is plane and the displacements of the particle in the x- and z-directions are independent, as described by (2.13). For brevity's sake we shall speak only of the x-motion, but the results apply equally to the z-motion along the binormal.

All previous results remain valid of course, but the fact that the motion is described now by a vector with two components only $\binom{x}{x'}$ allows us to go a little further. In fact, as the transfer matrix has the property

$$\det \mathbf{M}_x = 1 ,$$

the characteristic equation will take the form

$$\lambda^2 - \lambda \operatorname{Sp} \mathbf{M}_x + 1 = 0 , \tag{3.19}$$

so that

$$\lambda_{1,2} = \tfrac{1}{2} \operatorname{Sp} \mathbf{M}_x \pm i \left[1 - \tfrac{1}{4} \operatorname{Sp}^2 \mathbf{M}_x\right]^{\frac{1}{2}} . \tag{3.20}$$

We see that the eigenvalues for both x- and z-components of the motion are complex conjugate and reciprocal. The stability condition turns out to be the inequality

$$|\operatorname{Sp} \mathbf{M}_x| = 2|\cos \mu_x| < 2 . \tag{3.21}$$

We note that in the case of degeneracy of the eigenvalues ($\lambda_1 = \lambda_2 = \pm 1$; $\operatorname{Sp} \mathbf{M}_x = \pm 2$) both stability and instability are possible. However this case does not present interest for accelerator theory and will not be considered further.

Up to now we have not made any assumptions about the shape of the function $g(\vartheta)$, so that eqs. (2.13) have the general form of Hill's equations [4], i.e. non-linear differential equations with periodic coefficients. To get a clearer view of the dependence of the stability of their solutions upon the form of the function $g(\vartheta)$ let us consider two cases which have been studied in detail.

(a) The elastic force coefficient is a harmonic of the azimuth

$$g(\vartheta) = g_0 + g_1 \cos N\vartheta , \tag{3.22}$$

so that each of the equations (2.13) has the form of the well-known Matthieu equation [4, 10, 11]. The basic results are shown in fig. 17 where have been shaded regions of parameter values where μ is real, i.e. where there is stability.

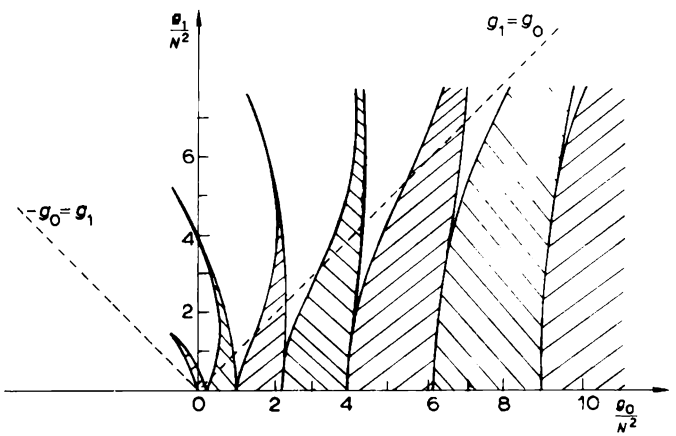

Fig. 17. Stability regions of Matthieu's equation. For $g_0 < -|g_1|$ the motion is always unstable. For $g_0 > |g_1|$ it is nearly always stable. In the region in between, there are relatively narrow zones of stability.

The characteristic property of Hill's equation, of which Matthieu's is a special case, is the existence of an infinite number of stability regions, separated by zones where μ is a complex. If the varying part g_1 of the elastic force tends towards zero, the motion is always stable when $g_0 > 0$ (case of the usual harmonic oscillator) and unstable when $g_0 < 0$. When $|g_1|$ grows, the motion may remain stable even when $g_0 < 0$, if $g_0 > -|g_1|$.

An interesting mechanical analogue exists, which illustrates the origin of such a kind of stability (see also ch. 1 § 4). The usual mathematical pendulum, when its point of suspension lies below the centre of gravity, will of course be unstable, as we have $g_0 < 0$, $g_1 = 0$. If, however, we move the point of suspension periodically in the vertical direction, the pendulum turned upside down may prove stable, inasmuch as its equation for small displacements has the form (2.13). The physical reason for this phenomenon, which has been observed experimentally [12], is that the time average of the restoring force acting on the pendulum

comes out positive, although this force changes sign in the course of a period of oscillation of the point of suspension (of course the parameters have to be chosen so that they correspond to one of the shaded areas in fig. 17). In the following paragraph we will show that an analogous situation is found in accelerators when the signs of g_x, g_z oscillate.

The existence of the oscillating part $g_1 \cos N\vartheta$ has also the consequence that there is a zone of instability for $g_0 > 0$, i.e. where the ordinary oscillator is stable. As the limits of the shaded zones correspond to the lines $\cos \mu = \pm 1$ they cut the abcissae axis at points given by

$$\frac{2\sqrt{g_0}}{N} = k\,, \qquad k = 1, 2, 3, \ldots \tag{3.23}$$

when going over to the ordinary pendulum ($g_1 \to 0$).

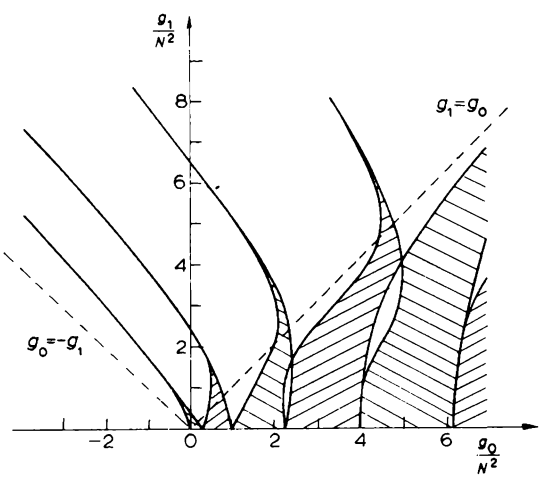

Fig. 18. The stability regions of Hill's equation for the case where function $g(\vartheta)$ is equal to constants in given intervals (see eq. 3.24).

(b) The situation is very nearly the same in the case of a Hill equation, where the function $g(\vartheta)$ has constant values in given intervals

$$g(\vartheta) = \begin{cases} g_0 + g_1 = \text{const.}, & 0 < \vartheta < \frac{1}{2}\vartheta_0 \\ g_0 - g_1 = \text{const.}, & \frac{1}{2}\vartheta_0 < \vartheta < \vartheta_0 \end{cases} \tag{3.24}$$

(a more general case is studied in ch. 7 § 2). The stability diagram has the same characteristic aspect (fig. 18). As before we can divide the area where the parameters vary into three characteristic regions.

(1) $g_0 < 0$, $g_1 < |g_0|$. The motion is always unstable, μ is complex.

(2) $g_1 > |g_0|$. The motion, as a rule, is unstable, but there are already narrow zones of stability where μ is real. For a given g_0 the width of the stability zone is greater, the smaller $|g_1|$.

(3) $g_0 > 0$, $g_1 < |g_0|$. The motion, as a rule, is stable, but there are relatively narrow zones of instability, beginning on the axis of abcissae at points given by (3.23). A peculiarity of this case of a step-wise $g(\vartheta)$ function is the appearance of 'islands' of instability for sufficiently large $|g_0|$, which do not exist with Matthieu's equation.

The features of the stability diagram we have noted remain the same for a rather wide class of functions $g(\vartheta)$ which appear in Hill's equation. As a consequence of this, betatron oscillations in various periodic systems have a series of properties in common, which we shall examine now.

4. BETATRON OSCILLATIONS IN PERIODIC MAGNETIC SYSTEMS

Up to now we have examined the stability criteria of the closed orbit only and left aside the character of the particles' motion around it.

Let us look again at system (2.14) and choose four linearly independent eigenvectors of transfer matrix $\mathbf{M}$ as initial conditions

$$X_i(0) = V_i \,. \tag{4.1}$$

These initial conditions determine four linear independent solutions unambiguously

$$X_i(\vartheta) = \mathbf{M}(0|\vartheta) V_i \,, \tag{4.2}$$

From this it follows immediately, for any integer value of k, that

$$X_i((k+1)\pi_0) = \lambda_i X_i(k\vartheta_0) = e^{i\mu_i} X_i(k\vartheta_0) \,. \tag{4.3}$$

In other words, four linearly independent solutions exist, each of which is multiplied by the constant number $e^{i\mu_i} = e^{i\nu_i\vartheta_0}$ when going through one whole period. This shows that it is possible to write down such solutions in the form

$$X_i(\vartheta) = e^{i\nu_i\vartheta} f_i(\vartheta) ; \qquad \nu_i = \frac{\mu_i}{\vartheta_0} = \frac{N\mu_i}{2\pi}, \tag{4.4}$$

where the $f_i(\vartheta)$ are vectors whose components depend on ϑ in a periodic way (with the magnet system period). Clearly the general solution will be a linear combination of terms as in (4.4). If the radial and vertical motions are uncoupled, it is convenient to take as linearly independent solutions simply complex conjugate ones, thus

$$\begin{aligned} x &= a_x f_x(\vartheta) e^{i\nu_x\vartheta} + \text{complex conjugate} \\ z &= a_z f_z(\vartheta) e^{i\nu_z\vartheta} + \text{complex conjugate} \end{aligned} \tag{4.5}$$

where $a_{x,z}$ are arbitrary complex constants and $f_x(\vartheta), f_z(\vartheta)$ periodic functions, determined by the structure.

Formulae (4.4) and correspondingly (4.5) are called 'Floquet's solution in the normal form' [4, 10, 11] and play an important role in accelerator theory. Functions $f_x(\vartheta)$ and $f_z(\vartheta)$, often called Floquet functions, are determined only to a constant factor for the time being. It is convenient to normalize them in such a way that

$$f(f^{*\prime} - i\nu f^*) - \text{compl. conj.} = -2i . \tag{4.6}$$

Evidently, this is always possible, as the left-hand side of (4.6) is simply equal to the imaginary part of the Wronskian of the solutions, which is an integral of motion, as was shown earlier. In the case of azimuthal symmetry, the Floquet functions become constants

$$f_x = \nu_x^{-\frac{1}{2}} = (1-n)^{-\frac{1}{2}}; \qquad f_z = \nu_z^{-\frac{1}{2}} = n^{-\frac{1}{2}} . \tag{4.7}$$

As we see, the oscillations around the closed orbit in a system chosen arbitrarily are not, in general, harmonic or even periodic. In the most general case, one can describe them with two periodic functions: a harmonic one with period $2\pi/\nu$ and a non-harmonic one with a period $\vartheta_0 = 2\pi/N$. It is natural to interpret the quantity ν as the frequency of betatron oscillations expressed in units of revolution frequency, thus generalizing the corresponding definitions for the azimuthally symmetric field. Thus the quantities ν and μ, which we introduced in a quite formal manner above, have a clear physical meaning: ν is the number of betatron oscillations performed during one revolution, and μ the phase

shift of the oscillation through one magnet period. Within the limits of the first stability region (on the left in figs. 17 and 18), μ varies from 0 to π and consequently ν from 0 to $\frac{1}{2}N$. In the second stability region μ varies from π to 2π and ν from $\frac{1}{2}N$ to N. In the third region μ varies from 2π to 3π and ν from N to $\frac{3}{2}N$ and so on. In the course of these frequency changes, the oscillation frequency may coincide with a multiple of the revolution frequency. However, such a choice of the parameters is to be avoided, as it leads to dangerous resonance phenomena (see ch. 3). At the boundaries between the regions of stability and instability, one finds that there is an integral or half-integral number of oscillations per magnet period.

Let us consider now the physical meaning of Floquet's function. We write solution (4.5) (i.e. for the x-oscillation) in the form

$$x = 2|a|\,|f(\vartheta)|\cos(\nu\vartheta + \chi(\vartheta) + \beta)\,, \tag{4.8}$$

where $|a|$ and β are arbitrary real constants and $\chi(\vartheta) = \arg f(\vartheta)$ is some periodic function. Then, if the motion is stable, $|f(\vartheta)|$ is the envelope for the form of oscillation considered and characterizes, for a certain value of $|a|$ depending on the initial conditions, the maximum displacement of the particle from the closed orbit as a function of azimuth. In particular, in an azimuthally symmetrical field $|f| = \text{const.}$ and the beam is uniform along the azimuth, as it must be as a consequence of symmetry. For arbitrarily chosen fields however, the transverse beam dimensions are proportional to $|f(\vartheta)|$ and depend on the azimuth in general. The quantity $|f(\vartheta)|$ goes through maxima and minima in each magnet period. At the same time according to (2.13), (4.5) and (4.6) this function satisfies the equation

$$|f(\vartheta)|'' + |f(\vartheta)|\,[g(\vartheta) - |f(\vartheta)|^{-4}] = 0\,. \tag{4.9}$$

As the derivative $|f|''$ has to be negative at the points where $|f|$ has a maximum, we arrive at the conclusion that the envelope of betatron oscillations has its maxima where $g(\vartheta) > 0$. This happens in the sectors of the magnetic period where the quasi-elastic force brings the particle back to the closed orbit. We will call such sectors focusing ones. It is just this property which provides the possibility of focusing with systems where the function $g(\vartheta)$ changes sign (strong focusing). In fact the particle wanders much further away from the orbit in focusing sectors ($g > 0$)

than in defocusing ones (we suppose of course that the motion is stable). Owing to this circumstance, the focusing action of the quasi-elastic force is greater than the defocusing one, which accounts for a resultant focusing effect. The typical form of the envelope and of two trajectories for a function $g(\vartheta)$ given by (3.24) is shown in fig. 19.

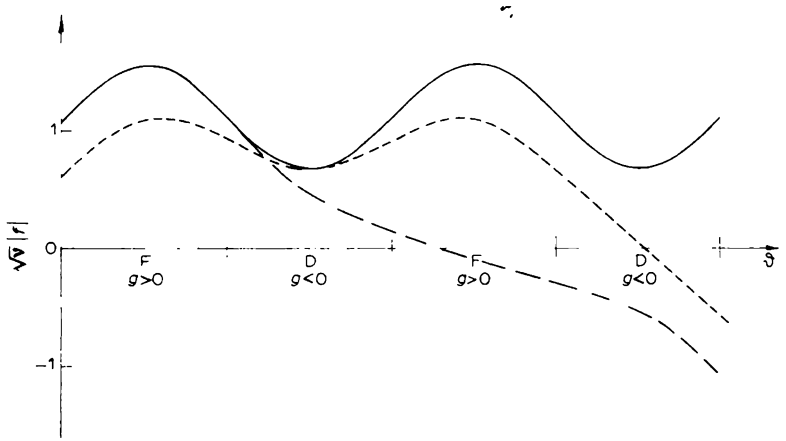

Fig. 19. Two possible particle trajectories and envelope of betatron oscillations (continuous line); F: focusing sector ($g>0$), D: defocusing sector ($g<0$).

Formulae (4.7) show a simple relation between Floquet's function and oscillation frequency in an azimuthally symmetrical system. One may extend these expressions to make them valid quite generally. We obtain in fact by writing Floquet's function in the form $f=|f|\,e^{i\chi}$ and using the normalizing condition (4.6)

$$\chi'(\vartheta)+\nu=|f|^{-2}\,, \tag{4.10}$$

from which follows

$$\nu=\langle|f|^{-2}\rangle=\frac{1}{\vartheta_0}\int_0^{\vartheta_0}|f|^{-2}\,d\vartheta\,. \tag{4.11}$$

We may consider (4.11) as an unambiguous definition of the frequency ν

(the equation $\cos\mu = \cos\nu\vartheta_0 = \frac{1}{2}\,\mathrm{Sp}\,\mathbf{M}$ defines ν apart from an added multiple of N).

It is possible to write the constant $2|a|$ in (4.8), which depends on the initial conditions, in a form analogous to the expression for the oscillation amplitude of the usual oscillation [13]

$$2|a|\,|f(\vartheta_i)| = \left[x_i^2 + \frac{R_0^2}{[\nu+\chi'(\vartheta_i)]^2}(\gamma_i-\gamma_{i0})^2\right]^{\frac{1}{2}}, \tag{4.12}$$

where

$$\gamma_{i0} = \frac{x_i}{R_0}\frac{|f(\vartheta_i)|'}{|f(\vartheta_i)|}. \tag{4.13}$$

Here $R_0 = K_0^{-1}$ and γ_i and x_i are the initial angle and co-ordinate of the path of the particle. As (4.12) shows, for a given x_i one gets the smallest maximum deviation from the orbit for a certain 'optimum' angle of entry γ_{i0}, determined by (4.13) and proportional in particular to x_i.

One often uses functions $P_x(\vartheta,\vartheta_i)$ and $P_\gamma(\vartheta,\vartheta_i)$ to characterize the betatron oscillations

$$P_x(\vartheta,\vartheta_i) = \frac{|f(\vartheta)|}{|f(\vartheta_i)|}\,; \qquad P_\gamma(\vartheta,\vartheta_i) = |f(\vartheta)|\,|f(\vartheta_i)|\,R_0\,. \tag{4.14}$$

Quantity $P_x(\vartheta,\vartheta_i)$ shows how many times the maximum deviation $2|a|\,|f(\vartheta)|$ is greater than the initial co-ordinate x_i on the assumption that the entry at the initial point ϑ_i occurred at the angle γ_{i0}. Function $P_\gamma(\vartheta,\vartheta_i)$, when multiplied with the initial entry angle γ_i, gives the maximum deviation at a point ϑ for the case where the particle entered without displacement from the orbit, i.e. for $x_i = 0$.

Let us now compare the efficiency of the various focusing methods. We notice that, for given initial conditions x_i and γ_i, the maximum displacement of the particle from the orbit is proportional to the quantity $|f|_{\max}$ which may be written

$$|f|_{\max} = \nu^{-\frac{1}{2}}F(\mu) \tag{4.15}$$

The function $F(\mu)$, sometimes called form factor, is in general not very far from 1, if the parameters are well chosen, and depend little on μ. So the focusing efficiency is basically determined by the frequency of

betatron oscillations ν. In strong-focusing magnets ν may be made substantially larger than in weak-focusing ones. We have then, for $\mu = \frac{1}{2}\pi$, $\nu = \frac{1}{4}N$, whereas $\nu \lesssim 1$ with weak focusing. This shows that the maximum displacements from the orbit diminish as $2n^{-\frac{1}{4}}$, a circumstance which has brought about the name 'strong focusing'.

We must now keep in mind that, if we make ν larger for given N, we will arrive eventually at the limit of the stability region (for $\nu = \frac{1}{2}N$),

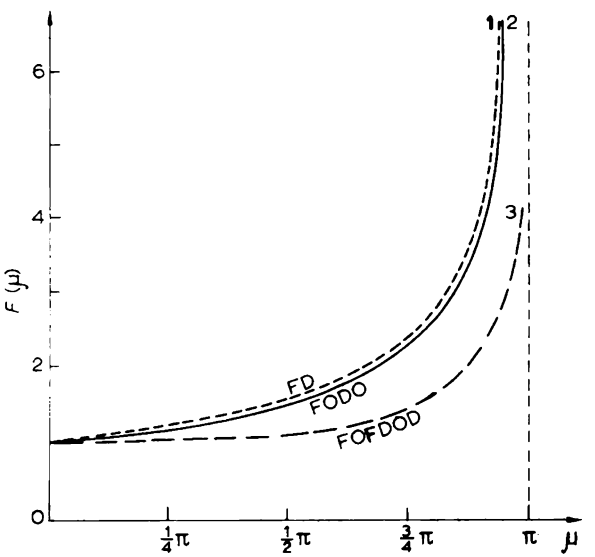

Fig. 20. Oscillation form factor in a magnetic system of the form (3.24) (curve 1). Curves 2 and 3 have been computed for a modification of this system, obtained by introducing straight sections with a relative length of 30%. The meaning of the indications FD, FODO, FOFDOD will be given in ch. 7 § 2.

near which the form factor $F(\mu)$ is growing substantially (fig. 20). In the second stability region and in the following ones, the quantity F is greater than in the first region as a rule. As we have $F^2 = |f|^2_{\max}\langle |f|^{-2}\rangle$, the envelope will be more and more distorted as we approach the limit of the stability region where the transverse beam dimensions begin to depend strongly on the azimuth. The smallest value $F = 1$ is obtained

on the contrary for $v \to 0$, when the oscillation has a nearly sinusoidal form, as we will see later.

We must emphasize that focusing must be ensured in both directions simultaneously in an accelerator. In addition, the quantities g_x and g_z are strongly tied by the relation (see 2.13)

$$g_x + g_z = \frac{K^2}{K_0^2}. \tag{4.16}$$

We have in general $|n| \gg 1$ on the greater part of the orbit in strong-focusing machines, so that $g_x \approx -g_z$. The stability regions for the z-motion will then be, according to figs. 17 and 18, mirror images of the

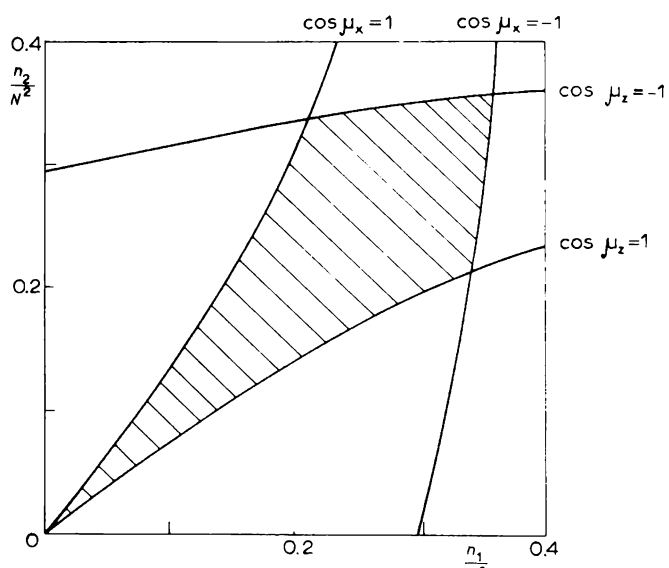

Fig. 21. Common stability region for x- and z-oscillations in a periodic system consisting of magnet blocks of equal length with alternating field indices equal to $-n_1$ and n_2 (with $n_{1,2} \gg 1$). The function $g(\vartheta)$ has in this case the form (3.24), where

$$g_0 + g_1 = \pm n_1, \qquad g_0 - g_1 = \mp n_2$$

for the x- and z-oscillations respectively.

x-stability regions with respect to the ordinates axis. It is easy to see that stability in both directions is achieved only in certain parts of the diagram, the largest of which has the characteristic neck-tie form (see fig. 21). This largest common stability region is the one used as a rule for strong-focusing accelerators, as it is less critical than the others with respect to the small unavoidable differences which occur between the real parameters of the machine and the computed ones. The use of the other common regions, although not excluded in principle, leads to great technical difficulties.

5 CANONICAL FORM OF THE EQUATION FOR BETATRON OSCILLATIONS

The absence of dissipative terms in the equations of motion (2.14), and as a consequence the property $\det \mathbf{M} = 1$, are not due to chance, nor are they linked to some assumption about the form of the magnetic field. This can be clearly shown by using the initial equations in canonical form. This form of writing appears to be extremely appropriate for investigating other questions of accelerator theory as well because it makes it possible to approach them from a general point of view. Hamilton's formalism is well suited, in particular, for studying strongly coupled oscillations, resonances and non-linear phenomena (ch. 3). In this connection it is useful to examine the general results obtained above from this point of view [1].

In the absence of electric field, Hamilton's function has the well-known form

$$\mathscr{H} = c\left[m_0^2 c^2 + \left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}\right)^2\right]^{\frac{1}{2}}, \tag{5.1}$$

where m_0 is the rest mass, $\boldsymbol{A}$ the time-independent vector potential of the magnetic field, and $\boldsymbol{P}$ the vector of the generalized momentum. To write the canonical equations in the natural system of co-ordinates, one has to define the momenta which are canonically conjugated to the generalized co-ordinates ϑ, x, z introduced above. Let us consider the passage to these co-ordinates (say from the usual cartesian co-ordinates $U^i = X, Y, Z$) as a canonical transformation. As we know, each canonical transformation going from some co-ordinates U^i and their canonical conjugate momenta $\boldsymbol{P}_i$ to new canonical variables u^k, p_k is achieved by a generating function F. If F is chosen in such a way that it depends on the old momenta $\boldsymbol{P}_i$ and the new co-ordinates u^k, the transformation will have the form [14]

$$\begin{aligned} p_k &= \frac{\partial F(P_i, u^k, t)}{\partial u^k}; \qquad U^i = -\frac{\partial F(P_i, u^k, t)}{\partial P_i}, \\ \mathscr{H}(p_k, u^k, t) &= \mathscr{H}(P_i(p_k, u^k), \quad U^i(p_k u^k), t) + \frac{\partial F}{\partial t}\bigg|_{P_i(p_k, u^k)}. \end{aligned} \tag{5.2}$$

In our case it is convenient to chose the generating function in the form

$$F = \boldsymbol{P}\boldsymbol{r} = \boldsymbol{P}[\boldsymbol{r}_0(\vartheta) + x\boldsymbol{n}(\vartheta) + z\boldsymbol{b}(\vartheta)] \; ; \quad u^1 = x \, , \quad u^2 = \vartheta \, , \quad u^3 = z \, . \tag{5.3}$$

Then, according to the first equation (5.2), the following momenta are conjugate with the co-ordinates ϑ, x and z

$$p_2 = \frac{\boldsymbol{P}}{K_0}\left[\tau(1+Kx) + \kappa(z\boldsymbol{n} - x\boldsymbol{b})\right] , \quad p_1 = \boldsymbol{P}\boldsymbol{n} \; ; \quad p_3 = \boldsymbol{P}\boldsymbol{b} \, . \tag{5.4}$$

The vector potential components are thus

$$A_1 = \boldsymbol{A}\boldsymbol{n} \; ; \quad A_2 = \frac{\boldsymbol{A}}{K_0}\left[\tau(1+Kx) + \kappa(z\boldsymbol{n} - x\boldsymbol{b})\right] \; ; \quad A_3 = \boldsymbol{A}\boldsymbol{b} \, . \tag{5.5}$$

It should be noted that p_2 and A_2 are not the tangential components* of the corresponding vectors $\boldsymbol{P}_\vartheta$ and $\boldsymbol{A}_\vartheta$. In the new variables Hamilton's function can be written

$$\mathscr{H}(p_1, p_2, p_3, \vartheta, x, z) = c\left\{m_0^2c^2 + \frac{1}{(1+Kx)^2}\left[K_0\left(p_2 - \frac{e}{c}A_2\right) + \kappa x\left(p_3 - \frac{e}{c}A_3\right) - \kappa z\left(p_1 - \frac{e}{c}A_1\right)\right]^2 + \left(p_1 - \frac{e}{c}A_1\right)^2 + \left(p_3 - \frac{e}{c}A_3\right)^2\right\}^{\frac{1}{2}} . \tag{5.6}$$

With the help of one of the canonical equations

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \frac{\partial\mathscr{H}}{\partial p_2} \tag{5.7}$$

we can convert to the azimuth ϑ as to a new independent variable. The equations of motion will then take the canonical form, with a new Hamiltonian

$$\mathscr{H}^* = -p_2(\mathscr{H}, \vartheta, p_1, p_3, x, z) = -\frac{(1+Kx)}{K_0}\left[\frac{\mathscr{H}^2}{c^2} - m_0^2c^2 - \left(p_1 - \frac{e}{c}A_1\right)^2 - \left(p_3 - \frac{e}{c}A_3\right)^2\right]^{\frac{1}{2}} - \frac{e}{c}A_2 - \frac{\kappa z}{K_0}\left(p_1 - \frac{e}{c}A_1\right) + \frac{\kappa x}{K_0}\left(p_3 - \frac{e}{c}A_3\right) , \tag{5.8}$$

where (x, p_1), (z, p_3), $(t, -\mathscr{H})$ are pairs of canonical conjugates. As A does not depend explicitly on time, t will be a cyclic variable, that is to say it does not appear explicitly in $\mathscr{H}^*$, but the quantity $\mathscr{H}$ is an integral of the motion (integral of energy). For this reason the transformation we have made virtually reduces the number of degrees of freedom from three to two. However, the new Hamiltonian $\mathscr{H}^*$ is now a periodic function of the independent variable ϑ.

* In fact, these quantities are the covariant components of the vectors $\boldsymbol{P}$ and $\boldsymbol{A}$ in the curvilinear co-ordinate system, which do not coincide with the tangential components, when $K \neq 0$, $\kappa \neq 0$ (see app. 1).

One obtains the linearized equations of motion by expanding (5.8) in powers of x, z, p_1, p_3 and keeping terms up to the second order. One must keep in mind, when doing this, that $\boldsymbol{A}$ obeys Maxwell's equation in vacuum

$$\text{rot rot } \boldsymbol{A} = 0$$

which sets limits to the coefficients of the expansion of A_1, A_2 and A_3. Without giving the intermediate steps let us write down for reference purposes the final expression of the Hamiltonian quadratic form

$$\mathscr{H}^*(p_x, p_z, x, z, \vartheta) = \frac{1}{2K_0 p}[p_x^2 + p_z^2] + \frac{p}{2K_0}\left\{\left[K_0^2 g_x + \tfrac{1}{4}\left(\frac{eB_\vartheta}{pc} + 2\kappa\right)^2\right]x^2 + \right.$$

$$\left. + \left[K_0^2 g_z + \tfrac{1}{4}\left(\frac{eB_\vartheta}{pc} + 2\kappa\right)^2\right]z^2\right\} + \frac{e}{2cK_0}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)xz +$$

$$+ \frac{1}{2K_0}\left(\frac{eB_\vartheta}{pc} + 2\kappa\right)(p_z x - p_x z), \quad (5.9)$$

where g_x and g_z are determined by (2.15) and p is the absolute value of the particle momentum. It is easy to check that (2.14) is a consequence of (5.9).

Thus, one can write the linear equations as a canonical system

$$\frac{\mathrm{d}x}{\mathrm{d}\vartheta} = \frac{\partial \mathscr{H}^*}{\partial p_x}; \quad \frac{\mathrm{d}p_x}{\mathrm{d}\vartheta} = -\frac{\partial \mathscr{H}^*}{\partial x}; \quad \frac{\mathrm{d}z}{\mathrm{d}\vartheta} = \frac{\partial \mathscr{H}^*}{\partial p_z}; \quad \frac{\mathrm{d}p_z}{\mathrm{d}\vartheta} = -\frac{\partial \mathscr{H}^*}{\partial z}, \quad (5.10)$$

having four linearly independent solutions

$$\boldsymbol{X}_i(\vartheta) = \begin{pmatrix} x_i \\ p_{x_i} \\ z_i \\ p_{z_i} \end{pmatrix}, \qquad i = 1, 2, 3, 4. \quad (5.11)$$

(It should be noted that in general (5.11) does not coincide with (3.2) as x' and z' are not canonical momenta in general or proportional to them, except for plane orbits in the absence of a longitudinal field.) It was shown by Liapunov and Poincaré that the transfer matrix for canonical systems possesses the important property that its eigenvalues form reciprocal pairs. Thus, if some quantity λ is an eigenvalue of the matrix, then λ^{-1} is also a root of the characteristic equation (see [15], [16] and app. B). If we use this result, we arrive again at criterion (3.14). It is important to stress that although the transfer matrix will differ somewhat from the matrix $\mathbf{M}$ considered in § 3 (because the vector $\boldsymbol{X}(\vartheta)$ is now expressed by (5.11) and not (3.12)), their eigenvalues coincide as they depend only on the structure of the magnetic field. This fact can be proved by the same means as were used to show the independence of the eigenvalues from the choice of the origin of the azimuth (see § 3).

It is easy to obtain the stability criteria (3.14) and the other results of § 3 by making use of the Liapunov-Poincaré theorem. The principal advantage of canonical variables is the possibility of applying some theorems of mechanics, giving the maximum of generality when considering this or that question. We can quote as an example of such an approach the design of a channel to inject particles into a magnetic system ([18], [19]), the contents of which fall, however, outside the scope of this chapter.

6. ADIABATIC CHANGE OF THE PARAMETERS OF BETATRON OSCILLATIONS

We have already mentioned in the introduction to this chapter that the change of particle energy in accelerators occurs as a rule slowly (adiabatically) with respect to the motion along the closed orbit and also with respect to betatron oscillations. This means that one can neglect the increase in energy in the distance of one period, although the energy varies substantially during the whole acceleration cycle. Other parameters also may vary slowly during the acceleration cycle, e.g. the field value on the orbit, the field index n, etc. The azimuthal field distribution and hence the form of the orbit may vary slowly also, because of various effects, such as saturation of the ends of magnet sectors. All this has an influence on the frequency and amplitude of betatron oscillations and brings about small changes therein.

Let us first consider this question in the simplest case of azimuthally symmetric systems where betatron oscillations around the orbit are independent and purely harmonic. In other words, we have to examine the behaviour of the amplitude A of a harmonic oscillator, when its parameters (elastic coefficient k and mass m) vary slowly. The solution is well known as it is easy in the case of periodic motion to find a quantity which is adiabatically invariant, that is remains constant with small changes of the parameters of motion [20]. Such a quantity is the surface of the phase plane region which is limited by a closed trajectory in phase space, or

$$I = \frac{1}{2\pi} \oint p \, dq = \tfrac{1}{2} A^2 \sqrt{km}, \tag{6.1}$$

where q is the co-ordinate of the oscillator and p the canonically conjugate momentum. In an azimuthally symmetric accelerator, $km = e^2 B_z^2 v^2/c^2$, m being the particle mass, and the amplitudes of the radial

and axial oscillations vary according to the law

$$A_x = \text{const.}\ B_z^{-\frac{1}{4}}(1-n)^{-\frac{1}{4}}\ ; \qquad A_z = \text{const.}\ B_z^{-\frac{1}{4}} n^{-\frac{1}{4}}. \tag{6.2}$$

Hence an important conclusion for the technique of accelerators: when the magnetic field increases on the circular orbit, the betatron oscillations decrease as $B_z^{-\frac{1}{4}}$. In the case of a constant field (phasotron), it is principally the change of n that influences the damping. For typical accelerator parameters, the adiabatic damping reduces the oscillation amplitude by a factor 10–30. This counteracts the influence of the various perturbations considerably and does not permit the amplitude to increase to dangerous values. In machines with magnetic fields varying in time, the form and mean radius of the orbit remain practically the same during the whole acceleration period (betatron, synchrotron, synchrophasotron). In such a case the mean value of the field along the orbit changes as the total momentum of the particle, so that the damping varies as $p^{-\frac{1}{2}}$. In machines where the magnetic field depends very little on radius (cyclotron, phasotron) the adiabatic damping is much weaker, although the particle momentum increases.

In order to generalize this result for any periodic magnetic system, we shall use the so-called mean-value method (see appendix C). The betatron oscillation equation, according to (1.7) and (2.11a,b) has in general the form

$$(Mx')' + Mg(\vartheta)x = 0\,, \tag{6.3}$$

where the role of the variable mass is played by the quantity

$$M = m\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = K_0 p\,. \tag{6.4}$$

Let us look for a solution of (6.3) in the form (4.5). We find, with complex amplitudes a and a^* the equation

$$a' = -\frac{f^* M'}{2\mathrm{i}M}\left[a(f' + \mathrm{i}\nu f) + a^*(f^{*\prime} - \mathrm{i}\nu f^*)\mathrm{e}^{-2\mathrm{i}\nu\vartheta}\right]. \tag{6.5}$$

For small M'/M the rapidly oscillating term proportional to $\mathrm{e}^{-2\mathrm{i}\nu\vartheta}$ on the right does not give a systematic change in a. We may neglect it and arrive at

$$|a|^2 \propto \exp\left[-\frac{1}{2i}\int \frac{M'}{M}\left[f^*(f'+i\nu f)-\text{compl. conj.}\right]d\vartheta\right] =$$

$$= \exp\left[-\int \frac{M'}{M}\,d\vartheta\right] = \frac{\text{const.}}{M}. \quad (6.6)$$

From here, using the definition of M in (6.4), we see that the betatron oscillation amplitudes, for small variations of their parameters, vary according to the law

$$A = \text{const.}\,(K_0 p)^{-\frac{1}{4}}|f|_{\max} = \text{const.}\,\langle B_z\rangle^{-\frac{1}{4}}\,F(\mu)/\nu^{\frac{1}{2}}, \quad (6.7)$$

where $\langle B_z\rangle = -cpK_0/e$ is the mean value of the vertical component of the magnetic field along the orbit and $F(\mu)$ the form factor of the oscillation (see eq. 4.15). As the same considerations are valid for the z-motion, we arrive at the following conclusion: when the frequency ν of betatron oscillation is constant the transverse dimensions of the beam decrease as $\langle B_z\rangle^{-\frac{1}{4}}$ as the mean field on the orbit increases.

Because of the importance of this conclusion, we shall give here its proof by the general theorems of mechanics. Let us consider for simplicity one-dimensional radial betatron oscillations described by the Hamilton function

$$\mathcal{H}(p_x, x, \vartheta) = \frac{p_x^2}{2K_0 p} + \frac{pK_0}{2}\,g_x(\vartheta)x^2\,, \quad (6.8)$$

which may be obtained directly from (2.13) and from the Hamiltonian (5.9) in the absence of coupling terms. Let us go over now to the new canonical variables I, Q with the help of the generating function

$$F(x, Q, \vartheta) = \frac{K_0 p x^2}{2|f|^2}\,\text{ctg}\,[Q+\chi(\vartheta)] + \frac{|f|'pK_0 x^2}{2|f|}, \quad (6.9)$$

from which, after a few transformations (see for instance [14]), we obtain making use of (4.9)

$$\mathcal{H}(I, Q, \vartheta) = I\nu\,, \quad (6.10)$$

$$x = \left[\frac{2I}{K_0 p}\right]^{\frac{1}{2}}|f(\vartheta)|\,\sin\,[Q+\chi(\vartheta)]\,; \qquad Q = \nu\vartheta + \text{const.} \quad (6.11)$$

It follows from the form of the Hamiltonian (6.10) that the quantity I is the integral of action, which has the property of being adiabatically invariant [14, 21]. The relationship

between the oscillation amplitude and the quantity I is given immediately by (6.11). Since quantity I remains constant for slow changes of the parameters, the amplitude has to vary according to the law (6.7).

It should be kept in mind that our reasoning has been limited until now to ideal magnetic systems, that is that we have neglected completely the various perturbations acting on the particle. The transverse beam dimensions in a real machine may depend as well on the pressure of the residual gas in the chamber, the emission of radiation in electron accelerators and a lot of other factors.

Up to now we have spoken only of adiabatic variations of the oscillation amplitude, although a slow change of parameters will cause the frequencies ν_x and ν_y to vary in the course of acceleration, that is the working point will move about in the stability diagram (figs. 17 and 18). When taking into account the defects of the magnetic system, which are unavoidable in practice, one finds oneself faced with the necessity to limit these movements of the working point (see ch. 3), and this the more severely the stronger the focusing, that is the bigger is ν. In particular, in strong-focusing synchrophasotrons the synchrotron oscillations are accompanied by rather large orbit variations and consequently displacements of the working point. This is why synchrotron oscillations have to be kept small, which is one of the characteristic peculiarities of this type of machine. In strong-focusing fixed-field accelerators, the demand for constant betatron frequencies requires geometrical similarity of all orbits. The fulfillment of this condition presents great technical difficulties in general.

7. SOME METHODS OF COMPUTING BETATRON OSCILLATION FREQUENCIES

Although the stability conditions (3.14) or (3.21) have a quite general character, they do not indicate immediately whether the motion is stable in a given magnet system. Actually, to determine the eigenvalue λ, it is necessary to know the exact form of the transfer matrix $\mathbf{M}$, that is to solve the equation of motion (2.13) or, in the general case, (2.14). Although such a method is not to be excluded in principle, it can be applied rigorously only in a rather limited number of problems or where

it is possible to divide the orbit in sectors, in each of which the solution can be found. We shall see examples of this kind in ch. 7 where we shall examine practical magnet systems used in accelerators. In recent years increasing attention has been given to complicated magnets which do not lend themselves to relatively simple calculations, and it seems appropriate to outline a few numerical and approximate methods of computing the parameters of betatron oscillations.

To avoid excessive complication, let us consider one-dimensional motion, that is, look for the normal solution of Hill's equation

$$x'' + g(\vartheta)x = 0 . \tag{7.1}$$

The results we get can be generalized for the case of two-dimensional oscillations, which however one seldom meets in accelerator theory.

(a) *Numerical methods.* When the magnet system is quite arbitrary, numerical methods are the only precise way to find the betatron oscillation parameters. If the orbit form and consequently the function $g(\vartheta)$ are known, it is convenient to base the calculation on the following considerations. Let us assume we have found two particular solutions $x_1(\vartheta)$ and $x_2(\vartheta)$ of (7.1) satisfying the initial conditions

$$x_1(0) = 1 \; ; \qquad x_1'(0) = 0 \; ; \qquad x_2(0) = 0 \; ; \qquad x_2'(0) = 1 . \tag{7.2}$$

Writing down these solutions in Floquet's form and determining the constants a and a^* from the initial conditions (7.2) we obtain the solution in the form

$$\begin{aligned} x_1(\vartheta) &= -\frac{f^{*\prime}(0) - \mathrm{i}\nu f^*(0)}{2\mathrm{i}} f(\vartheta)\, \mathrm{e}^{\mathrm{i}\nu\vartheta} + \text{complex conjugate}, \\ x_2(\vartheta) &= \frac{f^*(0)}{2\mathrm{i}}\, \mathrm{e}^{\mathrm{i}\nu\vartheta} f(\vartheta) \qquad\qquad + \text{complex conjugate}. \end{aligned} \tag{7.3}$$

It is easy to see that the half sum of x_1 and x_2' is equal to $\cos\mu$ when $\vartheta = \vartheta_0$. In fact, using the periodicity function $f(\vartheta)$ and the normalization condition (4.6) we find

$$x_1(\vartheta_0) + x_2'(\vartheta_0) = \mathrm{e}^{\mathrm{i}\nu\vartheta_0} + \text{complex conjugate}$$

and

$$\cos\mu = \cos\nu\vartheta_0 = \tfrac{1}{2}[x_1(\vartheta_0) + x_2'(\vartheta_0)] . \tag{7.4}$$

So it is sufficient, in order to find ν, to integrate numerically eq. (7.1) in the interval (0 to ϑ_0).

One can also find the Floquet function if one knows the solution $x_1(\vartheta)$ and $x_2(\vartheta)$. Actually, as far as $x_1(\vartheta)$ and $x_2(\vartheta)$ are linearly independent, it is possible to represent the quantity $f(\vartheta)e^{i\nu\vartheta}$, which is also a solution, as a linear combination of both

$$f(\vartheta) = e^{-i\nu\vartheta}[b_1 x_1(\vartheta) + b_2 x_2(\vartheta)] . \tag{7.5}$$

The complex constants b_1 and b_2 have to be chosen in such a way that the function $f(\vartheta)$ and its derivative are periodic, and this leads to the following system

$$\begin{aligned} b_1[x_1(\vartheta_0) - e^{i\nu\vartheta_0}] + b_2 x_2(\vartheta_0) &= 0 , \\ b_1[i\nu e^{i\nu\vartheta_0} + x_1'(\vartheta_0) - i\nu x_1(\vartheta_0)] + b_2[x_2'(\vartheta_0) - i\nu x_2(\vartheta_0) - e^{i\nu\vartheta_0}] &= 0 . \end{aligned} \tag{7.6}$$

Let us remark here that the condition for which the system is solvable is just eq. (7.4). With (7.6) we obtain the following expression for Floquet's function

$$f(\vartheta) = e^{-i\nu\vartheta} \frac{b_1}{x_2(\vartheta_0)} [x_1(\vartheta)x_2(\vartheta_0) - x_1(\vartheta_0)x_2(\vartheta) + x_2(\vartheta)e^{i\nu\vartheta_0}] . \tag{7.7}$$

From the normalization condition (4.6) we get also

$$|b_1|^2 = \frac{x_2(\vartheta_0)}{\sin \mu} . \tag{7.8}$$

In this way the envelope of the betatron oscillations is determined unambiguously by the solutions $x_1(\vartheta)$ and $x_2(\vartheta)$

$$|f(\vartheta)| = [x_2(\vartheta_0) \sin \mu]^{-\frac{1}{2}} |x_1(\vartheta)x_2(\vartheta_0) + x_2(\vartheta)(e^{i\mu} - x_1(\vartheta_0))| . \tag{7.9}$$

The argument $\chi(\vartheta)$ of Floquet's function may be found from (7.9) and (4.10) in terms of an arbitrary constant.

(b) *The method of averaging.* To determine approximately the orbits and frequencies of betatron oscillations, it is extremely useful to use the 'method of averaging'*, developed and strictly founded in the works of Bogoliubov and Krilov [22] and of Bogoliubov and Mitropolsky [23].

* The basic concept of the method is given in appendix C.

Seen from a physics point of view, the averaging method will be applied to compute betatron oscillations mainly when their wavelength is large compared to the magnet period, i.e. when the focusing is not too strong. Under such conditions it is natural to expect that the motion will be determined principally by quasi-elastic forces averaged over one period, and that the structure of each element will manifest itself only in small deviations from the averaged motion.

To bring one basic equation (7.1) into the form which is convenient to apply the averaging method (see appendix C) let us consider the two-dimensional vector $\boldsymbol{X} = \binom{x}{x'}$ and introduce the new independent variable $\tau = N\vartheta$. Then we get

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau} = \frac{1}{N}\mathbf{G}\boldsymbol{X}\,; \qquad \mathbf{G} = \begin{pmatrix} 0 & 1 \\ -g(\tau) & 0 \end{pmatrix}. \tag{7.10}$$

We shall assume the value of N to be large (or the dimensions of the period to be small) and will look for a solution in the form of a series of the powers of N^{-1}. Considering that equation (7.10) is linear and using the scheme developed in appendix C, we find that vector $\boldsymbol{X}(\tau)$ may be expressed, up to terms in N^{-2} as

$$\boldsymbol{X}(\tau) = \left\{1 + \frac{1}{N}\tilde{\mathbf{G}} + \frac{1}{N^2}(\widetilde{\mathbf{G}\tilde{\mathbf{G}}} - \bar{\tilde{\mathbf{G}}}\langle\mathbf{G}\rangle) + \ldots\right\}\xi. \tag{7.11}$$

With the same precision $\xi(\tau)$ is determined by

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \frac{1}{N}\left\langle\mathbf{G} + \frac{1}{N}\mathbf{G}\tilde{\mathbf{G}} + \frac{1}{N^2}\mathbf{G}(\widetilde{\mathbf{G}\tilde{\mathbf{G}}} - \bar{\tilde{\mathbf{G}}}\langle\mathbf{G}\rangle) + \ldots\right\rangle\xi. \tag{7.12}$$

In these formulae $\tilde{\mathbf{G}}$, $\bar{\tilde{\mathbf{G}}}$ and $\langle\mathbf{G}\rangle$ represent matrices with the elements

$$(\tilde{G})_{ik} = \tilde{G}_{ik}\,; \qquad (\bar{\tilde{G}})_{ik} = \bar{\tilde{G}}_{ik}\,; \qquad \langle G\rangle_{ik} = \langle G_{ik}\rangle\,, \tag{7.13}$$

where the integral operator $\sim$ is defined in appendix C (see ch. 5).

With the expression (7.10) for $\mathbf{G}$ it is easy to obtain the characteristic equation corresponding to (7.12) in the following form

$$\nu^2 = \langle g\rangle + \frac{1}{N^2}\langle\tilde{g}^2\rangle + \ldots, \tag{7.14}$$

and the motion, in the same approximation, will be described by the formula

$$x = \text{const.}\, e^{i\nu\vartheta}\left[1 - \frac{1}{N^2}\tilde{g} + \ldots\right]. \tag{7.15}$$

Having found the sense of the integrating operator, we arrive at the following expressions for the oscillation frequencies and Floquet's function

$$\begin{aligned} \nu^2 &\approx \langle[\int (g-\langle g\rangle)\mathrm{d}\vartheta]^2 + g\rangle \ ; \\ f(\vartheta) &\approx \nu^{-\frac{1}{2}}[1 - \int\int (g-\langle g\rangle)\mathrm{d}\vartheta'\mathrm{d}\vartheta''] \, , \end{aligned} \tag{7.16}$$

where the integration constants have to be chosen so that the mean value of the corresponding integral becomes equal to zero.

As an example let us consider the stepped function $g(\vartheta)$ which has constant values at successive intervals, for which one can also find the exact solution (ch. 7 § 2).

$$g = \begin{cases} +n_1 \, , & -\pi/N < \vartheta < 0 \, , \\ -n_2 \, , & 0 < \vartheta < \pi/N \, , \end{cases} \qquad n_{1,2} \gg 1 \, . \tag{7.17}$$

Using the approximate expression (7.16), we obtain

$$\mu_{z,x}^2 \approx \pm 2(\varphi_1^2 - \varphi_2^2) + \tfrac{1}{12}(\varphi_1^2 + \varphi_2^2)^2 \ ; \qquad \varphi_{1,2}^2 = \pi^2 n_{1,2}/N^2 \, . \tag{7.18}$$

It should be noted that we obtain exactly the same expression when we expand the exact solution (2.16) of ch. 7 in powers of μ and $\varphi_{1,2}$ and keep terms of the order μ^4 and $\varphi_{1,2}^4$. The averaging method will give large errors in the neighbourhood of the boundary of the stable region which corresponds to $\cos\mu = -1$. It would have been easy to foresee this, as on this boundary $\mu = \pi$, which means that the wavelength is twice the length of the period. This line can be called only conventionally a boundary as the averaging method gives only real values of μ in its neighbourhood. However, for small values of μ, formula (7.18) gives good approximations. As a rough indication one can say that the method can be applied right up to the centre of the stability region ($\mu < \frac{1}{2}\pi$), where it gives errors of the order of $10^0/_0$.

In this manner the averaging method leads to simple formulae making it possible to determine betatron oscillation frequencies easily, when

function $g(\vartheta)$ is explicitly given. Its weakness is that the accuracy decreases rapidly when the focusing strength increases.

In many cases another kind of approximate calculation known as Hill's method can be useful.

(c) *Hill's method.* Let us assume that function $g(\vartheta)$ in (7.1) can be developed in a convergent Fourier series

$$g(\vartheta) = \sum_{k=-\infty}^{+\infty} G_k e^{ikN\vartheta}, \tag{7.19}$$

where the series $\Sigma_1^{\infty} |G_k|$ converges. This condition is satisfied by every continuous function $g(\vartheta)$.

We seek a solution of (7.1) in the form of a complex Fourier series

$$x = e^{iv\vartheta} \Sigma x_k e^{ikN\vartheta}. \tag{7.20}$$

Introducing this in (7.1) and equating the coefficients of the individual Fourier harmonics we arrive at a system of linear equations

$$-(v+kN)^2 x_k + \sum_{l=-\infty}^{+\infty} x_l G_{k-l} = 0 \; ; \qquad k = 0, \pm 1, \pm 2, \pm \ldots \tag{7.21}$$

Let us put aside in each equation (7.21) the term $G_0 - (v+kN)^2$ and write the condition for the system to have a solution, that is the characteristic equation from which the frequency v has to be determined

$$\Delta(v) = 0, \tag{7.22}$$

where the elements of the determinant $\Delta(v)$ are

$$\Delta_{kl} = \begin{cases} \dfrac{G_{k-l}}{G_0 - (v+kN)^2} & \text{for} \quad k \neq l, \\ 1 & \text{for} \quad k = l. \end{cases} \tag{7.23}$$

The determinant $\Delta(v)$ is clearly an even periodical function of v with a period N, which is analytical everywhere, except for two poles at the points $v = -kN \pm \sqrt{G_0}$. Further $\Delta(v) \to 1$ when $|v| \to \infty$. The conditions we have imposed on the function $g(\vartheta)$ ensure the absolute convergence of the infinite determinant $\Delta(v)$ everywhere with the exception of the points $v = -kN \pm \sqrt{G_0}$, and the legitimacy of all the operations accomplished.

Let us further construct the function

$$\Delta_1(\nu) = \Delta(\nu) + \delta\left[\operatorname{ctg}\frac{\pi}{N}(\nu + \sqrt{G_0}) - \operatorname{ctg}\frac{\pi}{N}(\nu - \sqrt{G_0})\right], \qquad (7.24)$$

whereby we choose the constant δ so that function $\Delta_1(\nu)$ will not have poles anywhere. This is evidently possible if the second term on the right-hand side has poles of the same order and at the same points as $\Delta(\nu)$. As the periodic function $\Delta_1(\nu)$ is analytical everywhere and tends towards 1 when $\operatorname{Im}\nu \to \pm\infty$, we arrive at the conclusion that*

$$\Delta_1(\nu) \equiv 1\,. \qquad (7.25)$$

Consequently

$$\delta = \frac{1-\Delta(0)}{2\operatorname{ctg}\dfrac{\pi}{N}\sqrt{G_0}}, \qquad (7.26)$$

and eq. (7.22) takes the form

$$\sin^2\tfrac{1}{2}\mu = \sin^2\frac{\pi\nu}{N} = \Delta(0)\sin^2\frac{\pi\sqrt{G_0}}{N}. \qquad (7.27)$$

The characteristic equation (7.27) is exact, however it cannot be used as it is in practice because $\Delta(0)$ is a complicated expression of the Fourier coefficients G_k

$$\Delta(0) = \det\left\|\frac{G_{k-l} - \delta_{kl}(kN)^2}{G_0 - (kN)^2}\right\|. \qquad (7.28)$$

The infinite determinant $\Delta(0)$ however may be computed approximately if its elements decrease rapidly enough when one moves away from the main diagonal, according to the assumptions we have made above. Let us suppose that the order of smallness of the coefficients G_k is determined by the index k. Then the expression for $\Delta(0)$ can be put into the following form with an exactitude up to and including terms of the fourth order [25].

* This relation is also called theorem of Liouville [24].

$$\Delta(0) = 1 + \frac{\pi |G_1|^2 \operatorname{ctg} \frac{\pi}{N} \sqrt{G_0}}{2\sqrt{G_0} N(N^2 - 4G_0)} + \frac{\pi}{8N\sqrt{G_0}} \operatorname{ctg} \frac{\pi}{N} \sqrt{G_0} \times$$

$$\times \left\{ \frac{|G_1|^4}{(N^2 - 4G_0)^2} \left[\frac{\pi \operatorname{ctg} \frac{2\pi}{N} \sqrt{G_0}}{N\sqrt{G_0}} - \frac{1}{2G_0} + \frac{4}{N^2 - 4G_0} + \frac{9}{4(N^2 - G_0)} \right] + \right.$$

$$\left. + \frac{3(G_1^2 G_{-2} + G_2 G_{-1}^2)}{(N^2 - G_0)(N^2 - 4G_0)} + \frac{|G_2|^2}{N^2 - G_0} \right\} + \ldots \qquad (7.29)$$

For cases where function $g(\vartheta)$ happens to be even, Hill found an expansion for $\Delta(0)$ with an exactitude up to terms of sixth order inclusively.

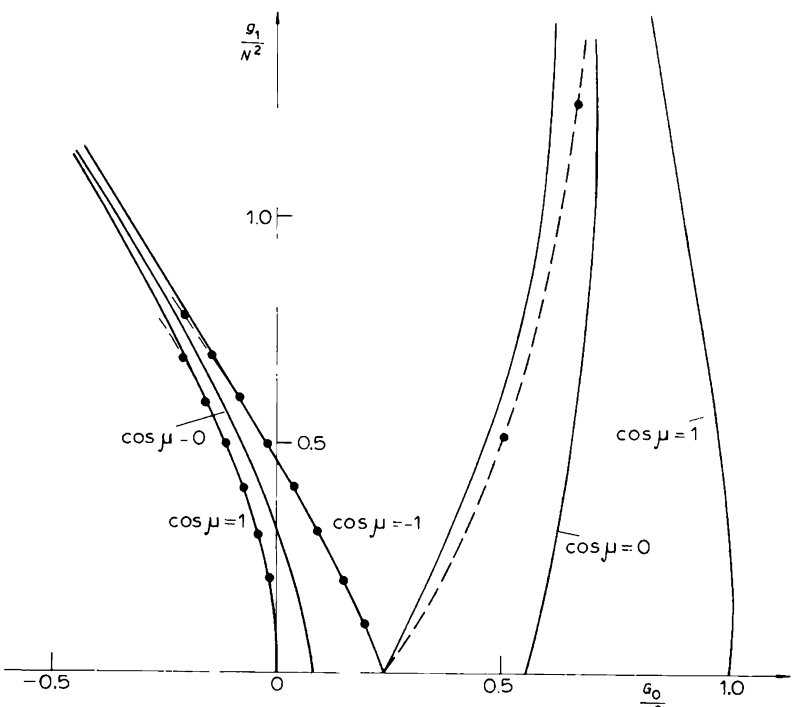

Fig. 22. First stability regions of Matthieu's equation. The full curve is the exact one, the dotted one is obtained from formula (7.29).

When using eq. (7.29) in practice, one should remember that it is only valid when the expansion coefficients G_k fall off rapidly enough with increasing index k.

In fig. 22 one can compare the exact computation with the approximate one in the case of Matthieu's equation, that is when function $g(\vartheta)$ in (7.1) has the form

$$g(\vartheta) = G_0 + g_1 \cos N\vartheta \; ; \qquad g_1 = 2|G_1| \, . \tag{7.30}$$

The coincidence of the two curves must be considered as relatively good, especially in the limits of the first stability region, which is generally used for accelerators (near the origin of the co-ordinates). In contrast with the previous method, Hill's method gives the domains of stability and of instability. The existence of only two stability regions in fig. 22 is caused by the fact that we have broken off expansion (7.29) after the terms of fourth order. When taking terms of sixth order into account a third stability region should appear, and so on.

In the first stability region Hill's method gives considerably better results than the averaging method, but requires more computation work. Moreover, because of the assumptions made, Hill's method cannot be applied formally in general to stepped functions $g(\vartheta)$ of the type (7.17) because in such cases the convergence of the determinant $\Delta(0)$ is not absolute.

There exist also other approximate methods to determine the frequency of betatron oscillations [1, 27, 28]. However they are to a large extent modifications of the two we have just considered, so we shall not consider them further.

CHAPTER 3

BETATRON OSCILLATIONS OF PARTICLES IN NON-IDEAL MAGNETIC FIELDS

As we have seen in the preceding chapter, knowledge of the magnetic field parameters and the characteristic orbits makes it possible, at least in principle, to determine if the orbits are stable, i.e. if the magnet system considered is adequate to guide the particles. However, even the most careful computation may be useless in practice because it is technically impossible to realize a magnetic field which corresponds exactly to the calculated one. For this reason it is natural to require that reasonably small imperfections of the magnetic system should cause only small perturbations in the motion of particles, which should as usual take place in the neighbourhood of the ideal orbit. In other words, stability should be conserved for sufficiently small perturbations. This introduces supplementary conditions for the choice of the magnet parameters and the need to realize them accurately during construction and operation of the accelerator. We shall show below how essential and difficult to observe these requirements are. The large dimensions of modern machines must be compatible with extreme precision of machining and assembly for the various parts, especially the electromagnet.

1. RESONANCES OF LINEAR BETATRON OSCILLATIONS IN NON-IDEAL MAGNETIC SYSTEMS

The results of the preceding chapter have a general character and apply to ideal magnet systems as well as to those which can be realized in practice. It should be stressed, however, that the main difference between them is that the number of periods is always less in the real system than in the ideal one, and is strictly speaking equal to one. Actually we have

assumed tacitly above that all N periods of the system were identical to each other, so that the magnetic field on the orbit repeats itself with a period $\vartheta_0 = 2\pi/N$. This is impossible to obtain in a real system in principle, as differences of manufacture in the magnet units will cause the field to be somewhat different in each one. Thus the properties of the system will only repeat after having accomplished one whole turn in it, that is with a period 2π. In other words the magnetic field can be written in the form

$$\boldsymbol{B}(\vartheta) = \boldsymbol{B}_0(\vartheta) + \boldsymbol{b}(\vartheta), \tag{1.1}$$

where $\boldsymbol{B}_0$ is the ideal magnetic field with period ϑ_0, and $\boldsymbol{b}$ is a small addition with period 2π*.

It is possible, with the results of the preceding chapter, to arrive at some general conclusions on the effect of small variations of the magnetic field on the stability of linear betatron oscillations. We shall consider a real system periodic with a period 2π. The transfer matrix in such a system has four eigenvalues λ_i' which must be close to the eigenvalues of the matrix for one revolution in the ideal system for sufficiently small perturbations, and form at the same time two pairs of mutually conjugate and two pairs of mutually inverse numbers. As one revolution covers N periods of the ideal system, we have $\lambda_i' \approx \lambda_i^N$. Let us assume that in the ideal system the stability conditions are realized, that is the quantities λ_i^N lie on the unit circle in the complex plane. If small changes in the parameters should lead to instability, that is to one of the configurations b, c, d in fig. 16, at least two eigenvalues λ_i^N must be near each other (as we say in this case, near to degeneracy). Let us see under what conditions this can take place.

We assume for simplicity that in the ideal system x- and z-motions are decoupled, so that the quantities λ_i^N take four values $\lambda_x^N, \lambda_x^{*N}, \lambda_z^N, \lambda_z^{*N}$, that is $e^{\pm 2\pi i \nu_x}, e^{\pm 2\pi i \nu_z}$. By equating these quantities two by two and remembering that complex exponents are equal when they differ by $2\pi i k (k = 0, \pm 1, \pm 2, \ldots)$ we arrive at the following important statement:

Small perturbations of the magnet system can destroy the stability of linear oscillations only in the case where the ideal system is such that one of the following equations is nearly fulfilled

* This symbol should not lead to confusion with the binormal vector $\boldsymbol{b}$, as the latter is not used in this chapter.

$$\nu_x = \tfrac{1}{2}k \; ; \quad \nu_z = \tfrac{1}{2}k \; ; \quad \nu_x \pm \nu_z = k \; ; \qquad k = 0, \pm 1, \pm 2, \ldots \quad (1.2)$$

or, more generally, if

$$n_x \nu_x + n_z \nu_z = k \, , \qquad |n_x| + |n_z| \leqslant 2 \quad (1.2a)$$

(n_x and n_z being integers). Physically this corresponds to one of the following cases*:

a) one revolution contains an integral or half-integral number of oscillations of one of the forms of oscillations (vertical or horizontal).

b) the sum or difference of the wave numbers of the two oscillations in one revolution is an integer.

Let us see now qualitatively to what physical phenomena the fulfilment of one of the eqs. (1.2), often called resonance conditions, lead. This name comes from the fact that if (1.2) is fulfilled, the magnetic field perturbations, which repeat after 2π, act on the betatron oscillation in a resonant manner, usually causing instability.

The mechanism of this interaction is particularly clear in the case of a simple or external resonance, when an integral number of betatron oscillations occurs in one revolution ($\nu = 1, 2, 3, \ldots$). The additional magnetic field $\boldsymbol{b}$ causes a Lorentz force to appear, which acts on the particle with a period 2π. The forced oscillations caused by this force are equivalent to the appearance of a new closed trajectory, or a perturbation of the closed orbit. In contrast to the ideal orbit, which has a period ϑ_0, this perturbed closed orbit has the same period as the perturbation, i.e. 2π.

In general, small disturbances in the magnetic field give rise to small deformations of the orbit, except in the resonant case, when an integral number of oscillations occurs in one revolution. The excursion away from the closed orbit then increases very rapidly and tends towards infinity exactly at resonance, as we have said already when discussing the azimuthally symmetrical field (ch. 1 § 4).

An analogous situation happens with any arbitrary magnetic system. If ν is an integer, the external force always acts in phase with the betatron

* The significance of these conditions lies in the fact that the stability regions of the real system differ somewhat from the stability regions of the original ideal system. In the framework of perturbation theory this difference leads to the appearance of resonance lines (bands), which will be examined later in this chapter.

oscillations so that there is no periodic solution in the linear case. The deviations of the orbit from the ideal one are then so large that considerations based on the linear equations of motion cannot apply.

According to (1.2), an instability of another type will be observed for $\nu = \frac{1}{2}k$, that is when a half-integral number of betatron oscillations are contained in one revolution ($\nu = \frac{1}{2}, 1, \frac{3}{2}, \ldots$). Let us write the transfer matrix in the real system in the form

$$\mathbf{M} = \mathbf{M}_0^N + \delta\mathbf{M} , \tag{1.3}$$

where $\mathbf{M}_0$ is the matrix of one period of the ideal system and matrix $\delta\mathbf{M}$ depends on the various perturbations having a period 2π. The quantity μ for the real system is then determined by the relation

$$\cos\mu = \tfrac{1}{2}\,\mathrm{Sp}\,\mathbf{M} = \cos\mu_0 N + \tfrac{1}{2}\,\mathrm{Sp}\,\delta\mathbf{M} . \tag{1.4}$$

Let us choose the parameters of the ideal system in such a way that the number of betatron oscillations per revolution will be near a half integer:

$$\mu_0 N = \pi k + \Delta , \qquad \Delta \ll 1 ,$$

so that

$$\cos\mu = (-1)^k(1 - \tfrac{1}{2}\Delta^2) + \tfrac{1}{2}\,\mathrm{Sp}\,\delta\mathbf{M} . \tag{1.5}$$

It is easy to see that if the 'distance' $\Delta^2 < \mathrm{Sp}\,\delta\mathbf{M}$ is sufficiently small we shall have, instead of stable oscillations, an exponential increase of the deviation from the orbit, which will be proportional to $\exp(\mathrm{Im}\,\nu)\vartheta = \exp\sqrt{\mathrm{Sp}\,\delta\mathbf{M} - \Delta^2}\,(\vartheta/\vartheta_0)$. We should stress that such an instability occurs even if the number of oscillations per turn is not exactly a half integer, but close to it ($\Delta \neq 0$).

It is important to note that the resonance we have just considered causes not a deformation of the closed orbit, but an increase of the amplitude of betatron oscillations. In the absence of the simple resonance mentioned above, it is usually called parametric or half-integral resonance.

One can get an idea of the mechanism of its action with the following mechanical analogue. Let us consider a pendulum with variable length

$$l = l_0 + l_1 \cos(\omega t + \psi)$$

and assume that the length variation is very small ($l_1 \ll l_0$) so that the

oscillations can be looked upon in a first approximation as harmonic with a frequency ω_0. Thus $\varphi = A \cos \omega_0 t$, where φ is the angular excursion from the equilibrium position. The pull on the shaft of the pendulum varies then with the double frequency

$$F_{\text{nat}} \sim \cos \varphi = 1 - \tfrac{1}{2}\varphi^2 = 1 - \tfrac{1}{4}A^2 - \tfrac{1}{4}A^2 \cos 2\omega_0 t\,, \tag{1.6}$$

and the work performed by the force stretching and compressing the shaft is proportional to the quantity

$$W = \int_0^t F_{\text{nat}} \frac{\mathrm{d}l}{\mathrm{d}t}\,\mathrm{d}t \sim A^2 \int_0^t \cos 2\omega_0 t \, \sin(\omega t + \psi)\,\mathrm{d}t. \tag{1.7}$$

In general the work $W(t)$ is a function of time which has a limit and changes sign, that is, an exchange of energy takes place between the oscillation and the source of the force. If we fulfil the condition of parametric resonance

$$2\omega_0 = \omega\,, \tag{1.8}$$

the work performed on the average by the force is different from zero and leads to an increase of the amplitude of oscillation. It should be noted that, depending on the initial conditions, the oscillations may be damped at the beginning but, as a more detailed investigation will show (see § 2), the amplitude will begin to grow after a finite interval of time. The physical interpretation of the resonance with the more general condition $2\omega_0 = k\omega$ (k an integer) is somewhat more complicated and requires consideration of higher harmonics of the oscillation.

From this example, the origin of the term parametric resonance appears clearly as the excitation of the oscillation is due to a variation of the oscillator parameters (in the case seen above the pendulum length) in resonance with the oscillation. The power received by the oscillator when a parametric resonance takes place is proportional to the square of the amplitude, so that the amplitude increases proportionally with time.

It is not difficult to apply this interpretation to the case of betatron oscillations in an accelerator. The resonances obtained here are evidently caused by the lower harmonics of the function $g_{x,z}(\vartheta)$ (see ch. 2, 2.13), that is by the deviations of the field index and the closed orbit curvature.

Because of these perturbations, betatron oscillations are tied to the longitudinal motion. Energy exchange between these degrees of freedom will also lead to a growth of oscillation amplitude. Depending on which of these degrees of freedom fulfils the resonance condition $\nu = \frac{1}{2}k$ the excursions will appear as radial or vertical oscillations.

If the x- and z-oscillations are coupled, the situation is more complicated. First it is clear that if the oscillation frequencies in the two directions are close enough, even small binding terms may cause an intense exchange of energy between them leading to the so-called 'difference resonance coupling' whose condition is

$$\nu_x - \nu_z = 0\,, \tag{1.9}$$

or more generally

$$\nu_x - \nu_z = k \text{ (integer)}\,. \tag{1.9a}$$

In the case of such a resonance the amplitude of either of the oscillations cannot grow without limits, as the sum of their energies remains constant. The situation is quite different in the case of the 'sum resonance coupling', where

$$\nu_x + \nu_z = k \text{ (integer)} \tag{1.10}$$

when the transverse degrees of freedom get coupled with the longitudinal motion, which makes possible an unlimited increase in amplitude in both directions of oscillation. In its nature, the sum resonance is somewhat analogous to the parametric resonance seen above, but it occurs when the oscillator has two degrees of freedom.

We shall see below that all the resonances we have enumerated, which are called linear resonances, are only special cases of more general resonance conditions which are characteristic of dynamical systems with three degrees of freedom (ϑ, x, z).

The existence of resonances has the result that the stability diagram appears crossed in several directions by forbidden bands, in the neighbourhood of the lines

$$\mu_{x,z} = \frac{\pi k}{N} \quad \text{and} \quad \mu_x \pm \mu_z = \frac{2\pi k}{N} \quad (k - \text{integer}) \tag{1.11}$$

These fields divide the whole stability region into individual cells, inside which the operating point must be chosen (fig. 23). At the same time

the dimensions of such a cell are smaller the greater N, that is the more complicated the magnet system. This leads to great difficulties when building strong focusing accelerators. In fact, if we require that during the whole acceleration cycle the parameters of the betatron oscillations

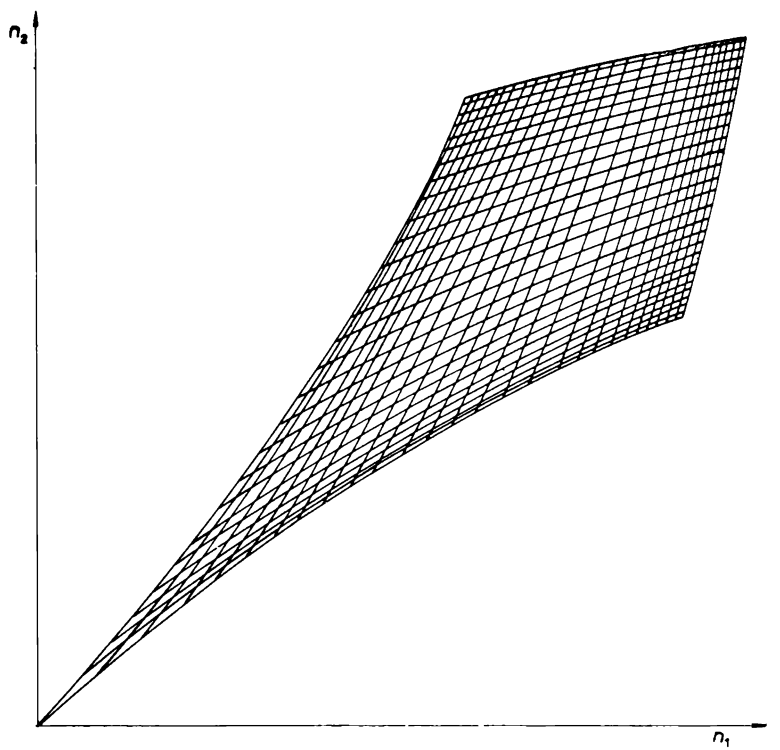

Fig. 23. Schematic representation of the stability region of a strong focusing accelerator, traversed by lines of simple and parametric resonances (see fig. 21).

should not take their resonance values, their frequencies must not vary by more than a quantity $\Delta\nu = \frac{1}{4}$ (in practice, even less). It is easy to see that for strong focusing accelerators, where ν is large, the relative tolerances on the magnet system parameters are much tighter than for weak focusing accelerators.

These simple physical reasonings cannot, of course, pretend to be rigorous and complete. On the contrary, the quantitative side of the question remains open: how dangerous are the resonances, what must be the tolerances of the magnet system, etc. These questions are extremely important for the design and operation of accelerators and a great number of papers by Soviet and foreign physicists are devoted to them (see for instance [1–9], and [11]). We have exposed the theory developed in these papers above in the context of pure linear oscillations. In the following sections, we shall dwell on a few complementary points which have been neglected above, and which are of interest in the neighbourhood of resonances.

2. LINEAR BETATRON OSCILLATIONS IN THE PRESENCE OF PERTURBATIONS

In ch. 2 we have obtained the linearized equations of betatron oscillations from the general eqs. (2.11) which are valid for deviations from any arbitrary closed curve. For such a curve we had chosen an orbit fulfilling in an ideal field the condition

$$B_x = 0\,, \qquad B_z = -\frac{pcK}{e}\,. \tag{2.1}$$

However, in a field with errors eq. (2.1) is not fulfilled for this curve, that is to say this curve does not give the equilibrium position for betatron oscillations. So it is natural to expect that the linearized equations for deviations from it will not be homogeneous in the non-ideal field.

To avoid useless complication, we shall assume throughout this chapter that the ideal field $\boldsymbol{B}_0$ has a symmetry plane in which lies also the initial orbit. We choose it as co-ordinate curve and linearize (2.11) of ch. 2, so that we get

$$\begin{aligned} &x'' + (g_x + \Delta g_x)x - \frac{e}{pcK_0^2}\frac{\partial b_z}{\partial z}\,z + \frac{eb_s}{pcK_0}\,z' = \frac{e}{pcK_0^2}\,b_z\,, \\ &z'' + (g_z + \Delta g_z)z + \frac{e}{pcK_0^2}\frac{\partial b_x(1+Kx)}{\partial x}\,x - \frac{eb_s}{pcK_0}\,x' = -\frac{e}{pcK_0^2}\,b_x\,, \end{aligned} \tag{2.2}$$

where the quantities g_x and g_z are related as before to the ideal motion and the quantities

$$\Delta g_x(\vartheta) = -\frac{K}{K_0^2}\left[\frac{1}{B_{0z}}\frac{\partial b_z(1+Kx)}{\partial x}\right]_{x=z=0},$$
$$\Delta g_z(\vartheta) = \frac{K}{K_0^2}\left[\frac{1}{B_{0z}}\frac{\partial b_x}{\partial z}\right]_{x=z=0}, \tag{2.3}$$

remain small. Further b_ϑ, b_x, b_z give the field perturbations and depend on the azimuth with a period 2π. As long as eqs. (2.2) are linear, their general solution is the superposition of forced oscillatory part, having a period 2π and describing the deviations of the real orbit from the ideal, and a part of free oscillations around it, determined by the homogeneous system in (2.2).

Let us first consider the question of the stability of the new closed orbit, i.e. consider the solution of system (2.2) without the terms on the right. As we assume that the perturbations of the field are small, it is convenient to apply the asymptotic method of averaging, which we have described briefly in ch. 2.

Because the system is quite near the ideal one, it is natural to seek a solution in a form comparable to (4.5) of ch. 2.

$$x = a_x f_x(\vartheta)\mathrm{e}^{\mathrm{i}\nu_x\vartheta} + \text{compl. conj.}\ ; \quad x' = a_x(f' + \mathrm{i}\nu f)_x \mathrm{e}^{\mathrm{i}\nu_x\vartheta} + \text{compl. conj.}$$
$$z = a_z f_z(\vartheta)\mathrm{e}^{\mathrm{i}\nu_z\vartheta} + \text{compl. conj.}\ ; \quad z' = a_z(f' + \mathrm{i}\nu f)_z \mathrm{e}^{\mathrm{i}\nu_z\vartheta} + \text{compl. conj.} \tag{2.4}$$

Here the complex quantities $a_{x,z}$, which are constant in the case of the ideal field, have now to be considered as functions of the azimuth, whereby their derivatives with respect to ϑ must be small to the same order as the perturbation. In fact, if we introduce the expressions (2.4) in the homogeneous system (2.2), we get

$$a_x' = \frac{f_x^*}{2\mathrm{i}}\frac{e}{pcK_0}\mathrm{e}^{-\mathrm{i}\nu_x\vartheta}\left[-\frac{pcK_0}{e}\Delta g_x \mathrm{e}^{\mathrm{i}\nu_x\vartheta} a_x f_x + \right.$$
$$\left. + \frac{1}{K_0}\frac{\partial b_z}{\partial z} a_z f_z \mathrm{e}^{\mathrm{i}\nu_z\vartheta} - b_\vartheta a_z (f' + \mathrm{i}\nu f)_z \mathrm{e}^{\mathrm{i}\nu_z\vartheta} + \text{c.c.}\right],$$

$$a_z' = \frac{f_z^*}{2i}\frac{e}{pcK_0}\,e^{-iv_z\vartheta}\left[-\frac{pcK_0}{e}\,\Delta g_z\,e^{iv_z\vartheta}a_z f_z - \right.$$

$$\left. - \frac{1}{K_0}\frac{\partial b_x(1+Kx)}{\partial x}\,a_x f_x e^{iv_x\vartheta} + b_\vartheta a_x (f' + ivf)_x e^{iv_x\vartheta} + \text{c.c.}\right]. \tag{2.5}$$

In accordance with the basic idea of the averaging method, the right-hand parts of these equations must be averaged with respect to the variable ϑ which enters explicitly. When this is done a series of particular cases can occur, which we shall now consider.

Non-resonant conditions. If none of the resonance conditions (1.2) is fulfilled, the right-hand part of eq. (2.5) contains rapidly oscillating terms which give an average of zero. The first terms in the square brackets, which do not depend in general upon the frequencies $v_{x,z}$, are an exception.

Far away from resonance the equations may be simplified as follows

$$a_x' = i\Delta v_x a_x\ ; \qquad a_z' = i\Delta v_z a_z\ , \tag{2.6}$$

where

$$\Delta v_{x,z} = \tfrac{1}{2}\langle |f_{x,z}|^2\,\Delta g_{x,z}\rangle\ . \tag{2.7}$$

The solution is elementary:

$$a_{x,z} = \text{const.}\,e^{i\Delta v_{x,z}\vartheta}\ . \tag{2.8}$$

If we compare this expression with formula (2.4) we arrive at the following conclusion: far from resonance, the perturbations of the magnetic field do not modify the stability regions for the betatron oscillations; they only change their frequencies by the quantity given in (2.7). Strictly speaking the form of the Floquet function changes also a little (there is a small additional term with period 2π) and small terms appear which oscillate with frequency v_z in the x-solution and vice-versa. In the majority of the cases however, these corrections are negligible*.

When the solution of the homogeneous system in (2.2) is known

* The frequency correction (2.7) appears not only in the non-resonant case, but also with resonances of any type. For this reason, we shall omit it in future in order to simplify the formulae, and we shall always designate by $v_{x,z}$ the betatron frequencies in the real system $v = v_0 + \Delta v$.

$$x \approx \text{const.}\, f_x\, e^{i\nu_x \vartheta} + \text{c.c.}\,, \qquad z \approx \text{const.}\, f_z\, e^{i\nu_z \vartheta} + \text{c.c.} \tag{2.9}$$

it is easy to find also the particular solutions $x_0(\vartheta)$ and $z_0(\vartheta)$ corresponding to the closed orbit. We can apply the simple method of the variation of constants. We require that the solution should be periodic

$$x_0(0) = x_0(2\pi)\,, \quad x_0'(0) = x_0'(2\pi)\,, \qquad z_0(0) = z_0(2\pi)\,, \quad z_0'(0) = z_0'(2\pi)\,, \tag{2.10}$$

and get for the deviation of the new closed orbit from the ideal one the expression

$$(x,z)_0 = \pm \frac{ie}{2pcK_0^2} \left\{ [1 - e^{-2\pi i \nu_{x,z}}]^{-1} \times \right.$$

$$\left. \times \int_{\vartheta}^{\vartheta + 2\pi} b_{z,x}(\vartheta')\, f_{x,z}(\vartheta)\, f_{x,z}^*(\vartheta')\, e^{i\nu_{x,z}(\vartheta - \vartheta')}\, d\vartheta' - \text{c.c.} \right\}. \tag{2.11}$$

We should note the important fact that the perturbation of the closed orbit, as a rule, will be greater, the larger the distortion of the median plane which causes it (see also ch. 1 § 4). This is especially noticeable when one is near to a simple (integral) resonance, when the characteristic resonance denominators $1 - e^{\pm i 2\pi\nu}$ in formula (2.11) become small. The same applies to the horizontal distortions of the orbit caused for instance by radial displacements of single magnet blocks.

In real systems, near an integer resonance, the conditions for parametric resonance are fulfilled as well, which makes the picture more intricate. For this reason, instead of analyzing their combined action, let us first examine separately the parametric resonance, which appears for half-integral values of the frequency ν.

The parametric, or half-integer resonance. Let us consider the behaviour of the amplitude of betatron oscillations in the neighbourhood of a parametric resonance, i.e. when condition

$$\nu = \tfrac{1}{2}k + \delta\,, \qquad \delta \ll 1\,, \quad (k \text{ integer}) \tag{2.12}$$

is fulfilled. We will drop the indices x, z in what follows, as all remarks apply to both forms of oscillation.

When (2.12) is satisfied, slowly varying terms appear in the right-hand parts of eqs. (2.5), which do not vanish when averaged. The simplified equations become

$$a' = Pa^* \, \mathrm{e}^{-2\mathrm{i}\delta\vartheta} , \tag{2.13}$$

where

$$P = \langle \tfrac{1}{2} \mathrm{i} f^{*2} \Delta g \, \mathrm{e}^{-\mathrm{i}\vartheta k} \rangle . \tag{2.14}$$

As usual, the behaviour of the absolute value of the complex amplitude a presents special interest, as it describes the maximum excursion A of the particle away from the orbit

$$A = 2|a| \, |f|_{\max} . \tag{2.15}$$

If we write $a(\vartheta)$ in the form

$$a(\vartheta) = |a| \, \mathrm{e}^{\mathrm{i}\eta} \tag{2.16}$$

and separate in (2.13) real and imaginary parts, we obtain the system

$$\begin{aligned} |a|' &= |a| \, |P| \cos(2\eta + 2\delta\vartheta - \arg P) , \\ \eta' &= -|P| \sin(2\eta + 2\delta\vartheta - \arg P) . \end{aligned} \tag{2.17}$$

The complete solution of these simplified equations may be found with quadratures, but will be expressed by elliptical integrals and be rather cumbersome. For our purpose, however, it is not so important to know the exact dependence of the oscillation amplitude on time so much as the limits of its variations, for which it is enough to study the phase trajectories in the plane of the variables $I = \frac{1}{2}(|a|^2)$ and $w = \eta + \delta\vartheta - -\frac{1}{2} \arg P$. This choice of variables is evidently not arbitrary. It is easy to see that the simplified eqs. (2.17) have here the canonical form

$$I' = -\frac{\partial \mathscr{H}}{\partial w} , \qquad w' = \frac{\partial \mathscr{H}}{\partial I} \tag{2.18}$$

with the Hamiltonian

$$\mathscr{H}(I, w) = I(\delta - |P| \sin 2w) . \tag{2.19}$$

As the Hamiltonian does not depend explicitly on time, $\mathscr{H}$ is an integral of motion, and gives at once the equation of the phase trajectories,

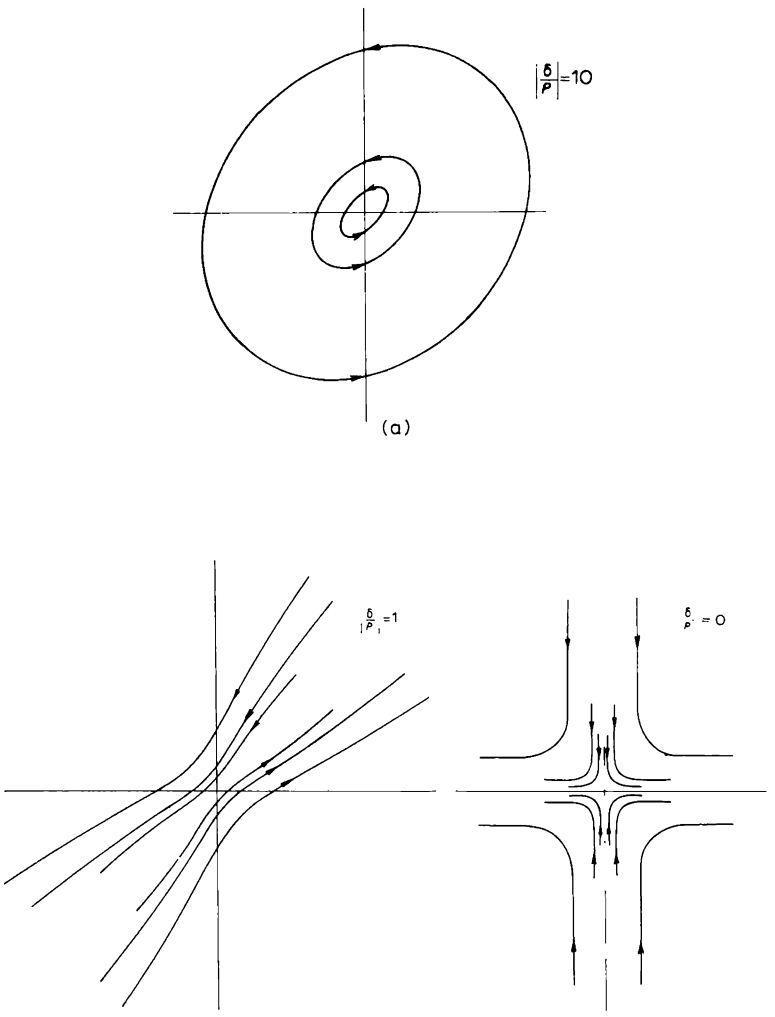

Fig. 24. Phase trajectories in the neighbourhood of a parametric resonance. The radius is proportional to the square of the amplitude, the azimuth is identical to the phase w.

which are the lines $\mathscr{H} = \text{const}$. In addition the canonical form makes it possible to apply some general theorems of mechanics to the case when the parameters of the oscillation vary slowly and also to survey the higher order resonances in a quite general manner.

Polar diagrams in phase-space describing the behaviour of the oscillation amplitude in the neighbourhood of a parametric resonance are shown in fig. 24. Each phase trajectory is a line $\mathscr{H} = \text{const}$. in polar co-ordinates, where the radius is proportional to the square of the amplitude I, and the azimuth represents the phase variable w. It is seen that the character of the phase diagrams depends to a large extent on the quantity δ/P. First, if this parameter is much larger than unity, that is if the working point lies far from the resonance or if the perturbation is sufficiently small, the phase trajectories are nearly circles, the oscillations take place with a practically constant amplitude, and a resonance does not appear (fig. 24a). Now let the parameter δ/P decrease towards unity. The phase trajectories will get distorted, displaying slow beatings of the amplitude, caused by the neighbourhood of the parametric resonance. It is easy to compute the depth of the beats, that is the ratio of maximum to minimum amplitude, by using eq. (2.19). As long as the motion takes place along a phase trajectory $\mathscr{H} = \text{const.}$, we have

$$\frac{|a|_{\max}}{|a|_{\min}} = \left[\frac{|\delta| + |P|}{|\delta| - |P|}\right]^{\frac{1}{2}}, \tag{2.20}$$

and the period of the beat pattern is

$$\Theta = \pi\left[\delta^2 - |P|^2\right]^{-\frac{1}{2}}. \tag{2.21}$$

Finally, for $|\delta| = |P|$ the beating depth and its period become infinite and the phase trajectories burst open. This shows that the oscillation amplitude grows indefinitely under these conditions, i.e. the parametric resonance proper (fig. 24b) takes place. If $|\delta|$ decreases further there is no substantial change in the picture, only the rate of increase of the amplitude becomes larger. Exactly at resonance, that is at $\delta = 0$, the amplitude, as we find easily from (2.17), varies at maximum rate

$$|a| \sim \exp(\pm |P| \vartheta) \tag{2.22}$$

and decreases or increases depending on the initial conditions (fig. 24c). We must stress that a decrease of the amplitude at resonance will change

inevitably into an increase, except for quite special initial conditions (corresponding formally to $\mathscr{H} = 0$). The increase of amplitude at a parametric resonance takes place very rapidly: thus a one percent perturbation Δg in a strong focusing accelerator causes an increase of the amplitude by a factor e after a few turns.

The considerations discussed above allow us to introduce the concept of stop bands, by which we understand the regions of the stability diagram where a resonance takes place.

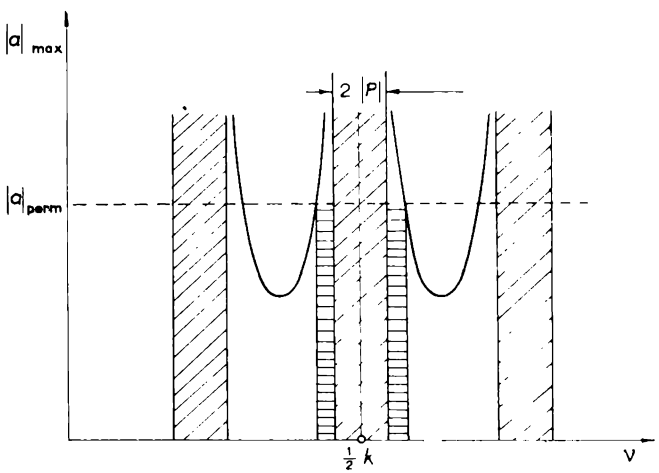

Fig. 25. Schematic representation of the regions of instability resulting from parametric resonance, $|a|_{\text{perm}}$ corresponds to the permissible oscillation amplitude.

This is illustrated in fig. 25, where together with the lines $\nu = \frac{1}{2}k$ we have drawn the lines $\delta = \pm |P|$, the oblique shading indicating the resonance regions. In practice, however, one should exclude the possibility of the working point falling not only in the resonance region, but also in its neighbourhood where the oscillation amplitude is subject to large beat patterns. Accordingly, we have marked with horizontal shading in fig. 25 forbidden field margins, the width of which depends on the permissible beating amplitude. It is clear that these bands should occupy only a small part of the working cell, so that one should certainly make the perturbation $|P|$ small. We shall dwell on these questions at length

in § 3, but let us only remark here that it is the k-th Fourier harmonic of the quantity $f^* \Delta g$ which is responsible for the excitation of the parametric resonance by $\nu = \frac{1}{2}k$, as follows from the definition of $|P|$ itself (see (2.14)).

Integer (simple) resonance together with a half-integer (parametric) one. When we studied motion without resonance, we noticed already that the deviation of the closed orbit from the ideal one has a tendency to become larger as the working point approaches the integer resonance line with $\nu \approx k$. If, however, we have also a parametric resonance associated with the integer one, solutions in the form (2.19) are, strictly, no longer valid. Thus, we have to make a special study in the immediate neighbourhood of such a resonance. We shall apply the method we used in the case of a pure half-integer resonance, but start, of course, from the inhomogeneous system (2.2).

When the resonance condition is satisfied

$$\nu_{x,z} = k + \delta\,, \qquad \delta \ll 1\,, \quad k - \text{integer} \tag{2.23}$$

the simplified equations for the complex amplitudes have the form

$$a'_{x,z} = P_{x,z} a^*_{x,z} e^{-2i\delta\vartheta} + G_{x,z} e^{-i\delta\vartheta}\,, \tag{2.24}$$

where

$$\begin{aligned} P_{x,z} &= \left\langle \frac{i f^{*2}_{x,z}}{2} \Delta g_{x,z} e^{-2ik\vartheta} \right\rangle, \\ G_{x,z} &= \mp \frac{ei}{2pcK_0^2} \langle f^*_{x,z} b_{z,x} e^{-ik\vartheta} \rangle\,. \end{aligned} \tag{2.25}$$

(We shall drop the indices x, z in the following.)

Separating real and imaginary parts in (2.24) and going over to the variables

$$I = \tfrac{1}{2}|a|^2 \quad \text{and} \quad w = \eta + \delta\vartheta - \arg G\,,$$

we obtain the equations

$$\begin{aligned} I' &= 2I|P| \cos\left(2w + \arg\frac{G^2}{P}\right) + \sqrt{2I}\,|G| \cos w\,, \\ w' &= \delta - |P| \sin\left(2w + \arg\frac{G^2}{P}\right) - \frac{|G|}{\sqrt{2I}} \sin w\,, \end{aligned} \tag{2.26}$$

which have again the canonical form with the Hamiltonian

$$\mathscr{H} = I\left(\delta - |P| \sin\left(2w + \arg\frac{G^2}{P}\right)\right) - \sqrt{2I}\,|G| \sin w\,. \tag{2.27}$$

It is important to note that eqs. (2.26) allow a stationary solution $I = I_0 = \text{const.}$, $w = w_0 = \text{const.}$, which describes the forced oscillations with constant amplitude

$$\begin{aligned} |a_0| &= \frac{|G|}{\delta^2 - |P|^2}\left[|P|^2 + \delta^2 - 2|P|\delta \sin\beta\right]^{\frac{1}{2}}, \\ \beta &= \arg\frac{G^2}{P}, \\ w_0 &= \operatorname{arctg}\frac{\delta - |P| \sin\beta}{|P| \cos\beta}. \end{aligned} \tag{2.28}$$

Depending upon the phase of the perturbation β, the stationary amplitude may have different values, lying however in the range $|a_0|_{\min} \leqslant |a_0| \leqslant |a_0|_{\max}$, where

$$|a_0|_{\max} = \frac{|G|}{|\delta| - |P|}, \qquad |a_0|_{\min} = \frac{|G|}{|\delta| + |P|}. \tag{2.29}$$

It is easy to understand the physical meaning of the stationary solution, if one notes that it corresponds to

$$\eta' = -\delta = \text{const.} \tag{2.30}$$

As the quantity η' represents a correction to the frequency we arrive at the conclusion, when comparing (2.30) with (2.4), that the stationary amplitude refers to oscillations performed with exactly the frequency of the external force. As the latter is periodic with a period 2π, the stationary solution evidently describes the perturbed closed orbit

$$(x, z)_0 \approx |a_0|\, f(\vartheta)\, \mathrm{e}^{\mathrm{i}k\vartheta} + \text{c.c.} \tag{2.31}$$

with a maximum deviation from the ideal orbit described by (2.29).

We should remark that the expressions we have obtained describe the motion sufficiently well in the neighbourhood of resonance only. At large distances from resonance it cannot, however, be used to determine

the deviation of the orbit. If the formulae of this section should apply far from resonance, and in particular be used with (2.11), approximations of a higher grade in the averaging method are essential.

The structure of the phase space diagrams near an integer resonance are rather similar to the previous case (fig. 26). In fact, with a simple change of variables

$$a \to a_0 + a$$

the terms describing the integral resonance vanish from the simplified equations, which become identical to system (2.17) describing the param-

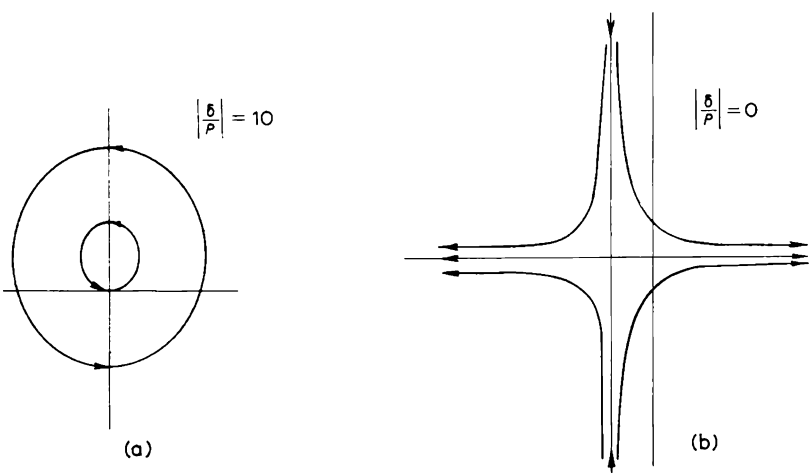

Fig. 26. Trajectories in phase space in the neighbourhood of an integral resonance appearing at the same time as a half-integral one.

etric resonance. This shows that the integral resonance does not influence the half-integral one, in the sense that the qualitative behaviour of the oscillation amplitude around the distorted orbit is determined by the parametric perturbation P. The contrary is not true in general as the form of the closed orbit itself depends on the strength of the parametric resonance. An interesting consequence of this is the fact that the distortion of the closed orbit becomes infinite not only exactly at the integer resonance, but also on the edge of the stop band of the

parametric resonance, for $\delta = \pm|P|$. The simultaneous action of the integral and half-integral resonance is shown in fig. 27, where the dependence of the quantity $|a_0|$ on ν is indicated, together with the unstable regions. According to (2.29) one can determine the limits of these stop bands by the quantities

$$\delta\nu_{\text{left}} = -|P|\,, \qquad \delta\nu_{\text{right}} = |P|\,. \tag{2.32}$$

The oscillation amplitude of the distorted closed orbit becomes infinite on the edge of these stop bands, which lie in the neighbourhood of

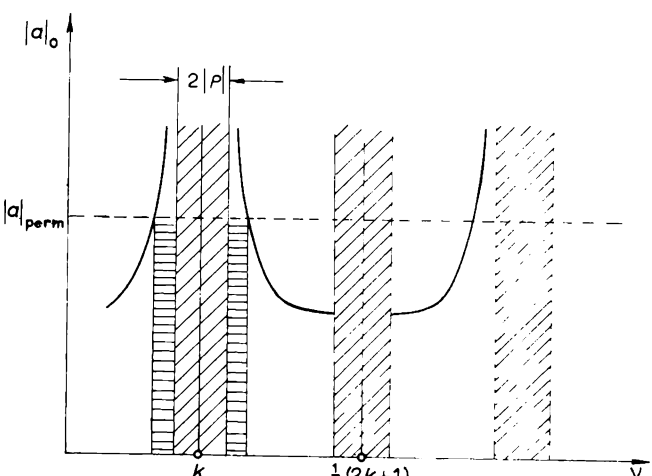

Fig. 27. Simultaneous action of an integral and half-integral resonance. The symbols are the same as in fig. 25.

integer ν values. If the maximum permissible value $|a|_{\text{perm}}$ that this distortion should not exceed is given, the corresponding forbidden band is wider just as in the case of the half-integral resonance.

Coupling resonances. Unlike ideal magnetic systems, where the x- and z-oscillations occur independently, as we have assumed above, real magnetic systems give a coupling between these two degrees of freedom. This can easily be seen already from the initial eqs. (2.2). As long as the coupling terms are small and the coupling resonance conditions (1.9a) or (1.10) are not fulfilled, the coupling cannot influence the motion

noticeably or lead to a systematic variation of the oscillation amplitude. This appears clearly in the simplified equations where the coupling terms between the x- and z-motions vanish when averaged, if one of the conditions

$$\nu_x \pm \nu_z = k + \delta\,, \qquad \delta \ll 1\,, \quad k - \text{integer} \tag{2.33}$$

is not fulfilled. Let us now consider these two cases separately.

a) *Sum resonance.* If we come back to eqs. (2.5) and average them with respect to ϑ, we easily find that in the neighbourhood of the sum resonance the simplified equations take the form

$$\begin{aligned} a_x' &= Q_z a_z^* \, \mathrm{e}^{-\mathrm{i}\delta\vartheta}\,, \\ a_z' &= Q_x a_x^* \, \mathrm{e}^{-\mathrm{i}\delta\vartheta}\,, \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} Q_z &= \frac{e}{2\mathrm{i}pcK_0} \left\langle f_x^* \left[\frac{1}{K_0} \frac{\partial b_z}{\partial z} f_z^* - b_\vartheta (f^{*\prime} - \mathrm{i}\nu f^*)_z \right] \mathrm{e}^{-\mathrm{i}k\vartheta} \right\rangle, \\ Q_x &= \frac{e}{2\mathrm{i}pcK_0} \left\langle f_z^* \left[b_\vartheta (f^{*\prime} - \mathrm{i}\nu f^*)_x - \frac{1}{K_0} \frac{\partial b_x (1 + Kx)}{\partial x} f_x^* \right] \mathrm{e}^{-\mathrm{i}k\vartheta} \right\rangle. \end{aligned} \tag{2.35}$$

We notice immediately that the quantities Q_z and Q_x are not independent, but are linked by Maxwell's equations. In fact, if we subtract from the first equation (2.35) the second, and observe that div $\boldsymbol{b} = 0$, we get

$$Q_z - Q_x = \frac{\mathrm{i}e}{2pcK_0} \left\langle \left[(f_x^* f_z^* b_\vartheta)' - \mathrm{i} b_\vartheta f_x^* f_z^* (\nu_x + \nu_z) \right] \mathrm{e}^{-\mathrm{i}k\vartheta} \right\rangle . \tag{2.36}$$

The averaging signs in (2.36) refer to an integration along the ideal orbit, which can be performed by parts if one takes into account the resonance condition (2.33)

$$Q_z - Q_x = \frac{e\delta}{2pcK_0} \left\langle b_\vartheta f_x^* f_z^* \, \mathrm{e}^{-\mathrm{i}k\vartheta} \right\rangle . \tag{2.37}$$

We notice that this quantity is small, of the second order, being proportional to the derivative of the perturbation with respect to the 'distance' δ which is small in the vicinity of the resonance. To a first approximation with the averaging method, one can put simply

$$Q_z = Q_x = Q\,. \tag{2.38}$$

Let us now go over to real variables, writing

$$a_x = |a_x| e^{i\eta_x}, \qquad a_z = |a_z| e^{i\eta_z}.$$

Then we have

$$\begin{aligned} |a_x|' &= |a_z|\,|Q| \cos w, \qquad \eta_x' = -|Q| \left| \frac{a_z}{a_x} \right| \sin w, \\ |a_z|' &= |a_x|\,|Q| \cos w, \qquad \eta_z' = -|Q| \left| \frac{a_x}{a_z} \right| \sin w, \end{aligned} \tag{2.39}$$

where

$$w = \eta_x + \eta_z + \delta\vartheta - \arg Q. \tag{2.40}$$

It is not difficult to see that (2.39) can be written in canonical form, if one goes over to the variables

$$I_x = \tfrac{1}{2}|a_x|^2, \qquad I_z = \tfrac{1}{2}|a_z|^2, \qquad w_{x,z} = \eta_{x,z} + \tfrac{1}{2}(\delta\vartheta - \arg Q). \tag{2.41}$$

The corresponding Hamiltonian function

$$\mathscr{H} = -2|Q|\sqrt{I_x I_z}\,\sin(w_x + w_z) + (I_x + I_z)\tfrac{1}{2}\delta \tag{2.42}$$

does not depend explicitly on ϑ. Therefore the lines $\mathscr{H} = \text{const.}$ are phase trajectories. This presentation, however, is not very obvious, as phase space is four-dimensional in the case considered. However, some general results will come out of the following considerations. First of all let us note that we are mainly concerned with the behaviour of the amplitudes $|a_x|$ and $|a_z|$ and not of the phases $w_{x,z}$. In other words, it is essential to know, not the whole trajectory in phase space, but only its projection on the plane (I_x, I_z). At the same time one can easily see that besides the Hamiltonian $\mathscr{H}$, the eqs. (2.39) also give us another integral of the motion

$$|a_x|^2 - |a_z|^2 = \text{const.} \tag{2.43}$$

or

$$I_x - I_z = \text{const.} \tag{2.44}$$

Physically this means that if an increase in amplitude occurs, it will take place in both directions of oscillation simultaneously. As regards the phase space trajectories, their projections on the plane (I_x, I_z) are

simply straight lines which all have the same slope, according to (2.44).

In order to determine under what conditions the motion is stable, let us note that $|\sin(w_x+w_z)|$ in (2.42) cannot be greater than one. Consequently the phase trajectories can lie only in the region

$$[(I_x+I_z)\tfrac{1}{2}\delta-\mathscr{H}]^2-4|Q|^2 I_x I_z \leqslant 0\,. \tag{2.45}$$

Depending on the value of the distance δ from resonance this region will have the form of one or other conic section. A simple investigation leads to the following results:

i) $\frac{1}{2}|\delta/Q|>1$. The motion is finite, i.e. the amplitude remains within given limits performing beats with a period

$$\Theta = \frac{2\pi}{[\delta^2-4|Q|^2]^{\frac{1}{2}}}\,. \tag{2.46}$$

One can determine the dependence of the beat amplitude on the initial conditions from the phase trajectories equation (see [6] for instance).

ii) If the distance δ is small enough and if the working point lies inside a stop band $|\delta|<|2Q|$, the oscillation amplitude in both directions will grow indefinitely. Thus both directions of oscillation are subject to a resonant buildup at the same time, whereby the difference between their amplitudes squared remains constant. Consequently, the sum resonance is dangerous and has to be avoided in practical devices.

b). *Difference resonance.* The situation is not the same in the case of a difference coupling resonance, that is when

$$\nu_x-\nu_z = k+\delta\,, \qquad \delta \ll 1\,, \quad k-\text{integer}\,. \tag{2.47}$$

We have already seen qualitatively in § 1 that we are dealing here only with a transmission of energy from one of the directions of oscillation to the other, so that an infinite increase of the amplitude cannot occur. In fact, the simplified equations will have, in the neighbourhood of a difference resonance, the form

$$\begin{aligned} a_x' &= Q_z a_z\, e^{-i\delta\vartheta}\,, \\ a_z' &= Q_x a_x\, e^{i\delta\vartheta}\,, \end{aligned} \tag{2.48}$$

as one can easily check. Here the coefficients Q_z and Q_x are somewhat different from (2.35)

$$Q_z = \frac{e}{2ipcK_0}\left\langle f_x^*\left[\frac{1}{K_0}\frac{\partial b_z}{\partial z}f_z - b_\vartheta(f'+ivf)_z\right]e^{-ik\vartheta}\right\rangle,$$
$$Q_x = \frac{e}{2ipcK_0}\left\langle f_z^*\left[b_\vartheta(f'+ivf)_x - \frac{1}{K_0}\frac{\partial b_x(1+Kx)}{\partial x}f_x\right]e^{ik\vartheta}\right\rangle, \tag{2.49}$$

although there exists as before a relation between them, which is similar to (2.38)

$$Q_z \approx -Q_x^* = Q\,. \tag{2.50}$$

We obtain now, instead of (2.39)

$$|a_x|' = |a_z|\,|Q|\cos w\,, \qquad \eta_x' = -|Q|\left|\frac{a_z}{a_x}\right|\sin w\,,$$

$$|a_z|' = -|a_x|\,|Q|\cos w\,, \qquad \eta_z' = -|Q|\left|\frac{a_x}{a_z}\right|\sin w\,.$$

It is easy to see that this system possesses a first integral

$$|a_x|^2 + |a_z|^2 = \text{const.}\,, \tag{2.51}$$

proving that the amplitude will remain limited even exactly on resonance. At any distance δ from it the amplitudes will perform beats with a period

$$\Theta = \frac{2\pi}{[\delta^2 + 4|Q|^2]^{\frac{1}{2}}}\,. \tag{2.52}$$

Far from resonance ($\delta^2 \gg 4|Q|^2$) the amplitude modulation due to the beating is extremely small and it is only in the immediate vicinity of the resonance that a full energy transfer from one of the directions of oscillation to the other can take place.

Thus the difference resonance is dangerous only under certain conditions, since such an energy transfer can be undesirable for additional reasons. In particular, if the particles have relatively plenty of room for their radial oscillations while their vertical motion is severely limited, the difference coupling resonance can present a danger. In some cases this resonance can even be of advantage, for instance during injection for oscillation-damping (see ch. 5) etc.

Our study of the difference resonance has been based on the assumption that the perturbations causing the coupling were small. One can however prove that this assumption is not essential and that the motion can remain stable even with strong coupling of the two oscillations.

Let us now review some results of this section. The fact that the practical magnetic field has a period which is larger than the ideal one causes all stability regions to be divided into cells by the lines

$$\nu_{x,z} = \tfrac{1}{2}k\,, \qquad \nu_x + \nu_z = k\,, \quad k - \text{integer}$$

Each such line is surrounded by a forbidden band, inside which the oscillations are unstable in at least one direction. For correct operation of the accelerator, one should arrange first that the width of these bands is much smaller than the dimensions of the cell, and second, that the working point is as far as possible from resonances in the real system. As the cell dimensions are proportional to N^{-1}, these requirements are more stringent the larger the number of periods.

3. SOME EQUATIONS RELATED TO THE CORRECTION OF THE MAGNETIC FIELD

The results of the preceding section give us the means to evaluate the tolerances for various parameters characterizing the magnetic field, on the basis of the linear theory. However, an optimum system of tolerances cannot be found uniquely but depends very much on the practical conditions such as the size of the machine, the technical possibilities, etc. The tolerances on one parameter can be eased if the requirements on others are tightened. At the same time, accelerators of certain types, for instance with ring-shaped or with undivided weak or strong focusing magnets require a rather special approach to tolerances and field correction systems. We should like to stress that in all practical cases, however varied, it has been found extremely useful to provide means for individually correcting, in quite a general manner, the most dangerous azimuthal harmonics of the field perturbation.

We have already stressed that perturbations of various types will influence particle motion in quite different ways. Thus an azimuthal field asymmetry gives rise to an excursion of the closed orbit in the radial direction, whereas deformation of the median plane causes excur-

sions in the vertical one. Parametric resonances are related to perturbations of the field index $n(\vartheta)$ and also azimuthal asymmetry. Coupling of the oscillations appears when the median plane is distorted or if there is a weak longitudinal field. In all cases the shape of the azimuthal dependence of the perturbation and not only its amplitude, is important.

We have already had an occasion to realize that the different harmonics of the perturbations have a different action on the motion, in the case of the azimuthally symmetrical field (ch. 1, 4.10, 4.11). We stressed that the main contribution to the orbit deviation came from those harmonics whose order was adjacent to the betatron oscillation frequency ν. The same is true for any arbitrary magnetic system, with the difference that instead of expanding the perturbation in a Fourier series, one must expand it in terms of the eigenfunctions of the Hill equation.

The eigenfunctions $y_k(\vartheta)$ are periodic solutions of the equation

$$y'' + g(\vartheta) y = \Lambda u(\vartheta) y\,, \tag{3.1}$$

with a period 2π. Here $u(\vartheta)$ is some function having a period ϑ_0. The quantity $\Lambda = \Lambda_k$ for which (3.1) has a non-trivial periodic solution, is called an eigenvalue. According to general theorems [12], to each function $u(\vartheta)$ there corresponds a complete system of orthogonal eigenfunctions, which it is convenient to normalize to unity

$$\int_0^{2\pi} y_k y_{k'}^* u(\vartheta)\,\mathrm{d}\vartheta = \delta_{kk'}\,. \tag{3.2}$$

Thus it becomes possible to represent any periodic perturbation in the form of an expansion in eigenfunctions.

Let us consider the deviation of the closed orbit from the ideal one we considered above (§ 2), which is a periodic solution of the equation

$$x_0'' + g_x(\vartheta) x_0 = \frac{e}{pcK_0^2}\, b_z(\vartheta) \tag{3.3}$$

e.g. for the x-motion. (We shall not consider here the action of the parametric resonance.) Expanding $b_z(\vartheta)/u(\vartheta)$ in a series of eigenfunctions

$$\frac{b_z(\vartheta)}{u(\vartheta)} = \sum_{k=-\infty}^{+\infty} b_z^{(k)} y_k(\vartheta)\,, \qquad b_z^{(k)} = \int_0^{2\pi} b_z(\vartheta) y_k^*(\vartheta)\,\mathrm{d}\vartheta\,, \tag{3.4}$$

we obtain the following expression for the perturbed orbit

$$x_0(\vartheta) = \frac{e}{pcK_0^2} \sum_{k=-\infty}^{+\infty} \frac{b_z^{(k)}}{\Lambda_k} y_k(\vartheta) . \tag{3.5}$$

It is clear that the main contribution to $x_0(\vartheta)$ comes from the harmonics of the expansion which correspond to the smaller Λ_k, so that if one of the eigenvalues is near zero an integer resonance takes place.

In practice the eigenvalues Λ_k depend on the choice of the 'weighting' function $u(\vartheta)$ which is to some extent arbitrary. If we choose this function in the form $u(\vartheta) = \nu^{-2}|f|^{-4}$, the expansion (3.5) takes a simple form, as eq. (3.1) has then the following normalized periodic solutions

$$y_k = \frac{|f(\vartheta)|\,\nu^{\frac{1}{2}}}{\sqrt{2\pi}} e^{ik\hat{\vartheta}} , \qquad \hat{\vartheta} = \vartheta + \frac{\chi(\vartheta)}{\nu} , \tag{3.6}$$

where $|f|$ and $\chi(\vartheta)$ are the absolute value and the argument of Floquet's function. The eigenvalues have now the simple form

$$\Lambda_k = \nu^2 - k^2 . \tag{3.7}$$

Of course, other choices of $u(\vartheta)$ and systems of eigenvalues are possible, and can be more convenient in some cases (see for instance [5, 6, 24].

Introducing (3.6) in (3.5) we obtain the following expression for the closed orbit distortion

$$x_0(\vartheta) = \frac{e}{pcK_0^2} \sum_{k=-\infty}^{+\infty} \frac{b_z^{(k)}}{\nu^2 - k^2} y_k(\vartheta) = \frac{e\nu^{\frac{1}{2}}|f(\vartheta)|}{pcK_0^2\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \frac{b_z^{(k)}}{\nu^2 - k^2} e^{ik\vartheta} . \tag{3.8}$$

It is seen that one must first of all correct the perturbation harmonics which are nearest to the betatron frequency, if one wants to make the closed orbit distortion smaller.

In an azimuthally symmetrical field, where the Floquet functions are simply equal to the constant $\nu^{-\frac{1}{2}}$, the expansion in terms of eigenfunctions coincides with the usual expansion in a trigonometrical Fourier series

$$b_z^{(k)} = \int_0^{2\pi} b_z(\vartheta) y_k^*(\vartheta)\,d\vartheta = b_{z(k)} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} b_z(\vartheta) e^{-ik\vartheta} d\vartheta . \tag{3.9}$$

(In the absence of expansion coefficients in terms of eigenfunctions, the Fourier coefficients will be given subscripts.)

As in this case $\nu < 1$, the correction concerns the first Fourier harmonic of the azimuthal asymmetry, and is applied by means of special coils or shims. In order to be able to correct the field perturbation whatever its phase, one uses two independent groups of such coils, azimuthally spaced by $\frac{1}{2}\pi$ (these groups are sometimes called 'sine' and 'cosine' coils). As we see from (3.8) the second and third azimuthal field harmonics have tolerances that are $(4-\nu^2)/(1-\nu^2)$ and $(9-\nu^2)\ (1-\nu^2)$ times larger than the first one. The situation is somewhat more complicated if the weak focusing magnet has straight sections (ch. 7 § 2).

In any arbitrary periodic magnet system, the requirements regarding the coefficients of the expansion in eigenfunctions are similar, in principle, to those existing for the Fourier harmonics in the case of rotational symmetry. When $\nu > 1$ there are, in general, two or more harmonics with orders near ν, ($k > \nu$ or $k < \nu$) which need careful correction. When the harmonic order is further away from resonance, the influence of the particular harmonic falls rapidly, as shown by formula (3.8).

In practice it will be more convenient to correct the Fourier harmonics than the harmonics in terms of eigenfunctions. It is easily seen that a resonant deformation of the orbit will be caused not only by the (kth $= \nu$th) Fourier harmonic, but also by the $(N \pm k)$th, $(2N \pm k)$th harmonics, etc. In fact, we can expand the following quantity in a Fourier series

$$\exp\left[\mathrm{i}\frac{k}{\nu}\chi(\vartheta)\right]\nu^{\frac{1}{2}}|f(\vartheta)| = \sum_{l=-\infty}^{+\infty} a_{k(l)}\,\mathrm{e}^{\mathrm{i}l\vartheta}\,. \tag{3.10}$$

Introducing (3.10) in (3.4) we obtain the expression

$$b_z^{(k)} = a_{k,(0)}b_{z,(k)} + a_{k,(N)}b_{z,(N+k)} + a_{k,(-N)}b_{z,(k-N)} + \ldots\,, \tag{3.11}$$

which, in agreement with (3.5), shows that the excitation due to one eigenfunction $y_k(\vartheta)$ is equivalent to an excitation by a whole particular spectrum of Fourier harmonics. However, if the oscillations are sufficiently smooth, that is Floquet's function remains close to its average value, the quantities $a_{k(l)}$ will decrease rapidly with increasing number l, and the expansion coefficients in terms of eigenfunctions will not be very different from the coefficients of the trigonometric series.

In periodic magnet systems one can in principle use the same field correction methods as with symmetry of revolution. In addition if there are many independent structural units, the correction of the harmonics may actually be done by small displacements of the individual magnet blocks in the radial and vertical directions. In fact, a small displacement of a block by a quantity $\Delta(x, z)$ will cause a distortion of the ideal orbit equal to

$$b_{z,x} \approx nB_{0z}K\Delta(x, z). \tag{3.12}$$

Moreover, it is possible to correct an asymmetry and perturbations giving rise to x- and z-oscillations by slightly turning the axis of a single block away from the beam axis, so that one has

$$b_x \approx B_{0z}\varphi\,, \qquad \frac{\partial b_x}{\partial x} \approx -nB_{0z}K\varphi\,, \tag{3.13}$$

where φ is the angle by which the unit was turned.

When designing the magnet of an accelerator, one will aim to get the optimum position of the working point as far as possible from dangerous resonances. Also one will try to arrange that the width of the stop bands is much less than the dimensions of the working cell.

Correction of the betatron frequencies can be carried out either by tuning the field index n with the help of special coils or shims, or in larger strong focusing machines by installing along the orbit a number of quadrupole lenses, with the help of which one can alter somewhat the function $g(\vartheta)$. In accelerators with time-varying magnetic fields saturation effects of the iron of the magnet yoke may appear at the end of the cycle, and lead to a small decrease in the absolute value of $|n|$. To compensate the resulting change in the betatron oscillation frequencies, one has to vary the current in the correcting coils as a function of time.

It is easy to realize that the width of a given stop band is determined by the corresponding harmonic of the expansion in terms of eigenfunctions. Thus, for example, the perturbation exciting the parametric resonance at $\nu \approx \frac{1}{2}k$ has, according to (2.14) the form

$$2|P| = \frac{1}{2\pi}\left|\int_0^{2\pi} f^{*2}\Delta g\, e^{-ik\vartheta}\,d\vartheta\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} |f|^2 e^{-ik\vartheta}\Delta g\,d\vartheta\right| = \frac{1}{\sqrt{2\pi\nu}}\left|(|f|\Delta g)^{(k)}\right|, \tag{3.14}$$

or, if we expand in trigonometric functions

$$2|P| = \frac{1}{\sqrt{2\pi}} |(f^{*2}\, \Delta g)_{(k)}| \,. \tag{3.14a}$$

Thus the parametric resonance by $\nu \approx \frac{1}{2}k$ is excited by the kth Fourier harmonic of the quantity $f^{*2}\, \Delta g$ or by the kth, $(N \pm k)$th, $(2N \pm k)$th... harmonics of the perturbation Δg. To compensate these harmonics special correcting networks must be foreseen.

The same considerations are valid for coupling resonances. According to (2.35) and (2.49) the perturbation which is exciting them can be written in the absence of a longitudinal magnetic field,

$$2|Q| = \frac{e}{pcK_0^2\sqrt{2\pi}} \left| \left(f_x^* f_z^* \frac{\partial b_z}{\partial z} \right)_{(k)} \right| = \frac{1}{\sqrt{2\pi} K_0} \left| \frac{1}{\langle B_{0z} \rangle} \left(f_x^* f_z^* \frac{\partial b_z}{\partial z} \right)_{(k)} \right| \tag{3.15}$$

for a sum resonance, and

$$2|Q| = \frac{1}{\sqrt{2\pi} K_0} \frac{1}{\langle B_{0z} \rangle} \left(f_x^* f_z^* \frac{\partial b_z}{\partial z} \right)_{(k)} \tag{3.15a}$$

for a difference resonance. (It should be remembered that the quantity $\partial b_z/\partial z$ on the ideal orbit comes from angular positioning errors of single magnet blocks.)

It is not difficult to estimate the permissible relative value of the resonance harmonics on the basis of the formulae given above, if one starts from a tolerable maximum level of beats. As was the case for the closed orbit, one must make a compromise between the requirements of a sufficiently intense beam and the desire not to make the technical questions too difficult.

Of great practical importance is the question of the tolerances to which the magnet system has to be machined. The choice of these tolerances is a difficult task. One must take into account the beam intensity required, technical and economical possibilities and the effect of these tolerances on other parts of the accelerator, in particular the radio frequency accelerating system.

To determine the tolerance, one may begin by computing the motion of the particle under various typical perturbations, and from the results find the tolerances required to get the expected beam intensity.

In some cases the tolerance calculations for strong focusing accelerators can be based on statistical considerations. Let us assume that we may divide the magnet system in a large number of sectors, in each of which the perturbations may be regarded as constant, whereas the perturbations in different sectors are statistically independent*. Such an assumption is not artificial as the magnet is usually designed in the form of separate units whose parameters in fact are scattered. Moreover, when placing these units into position it is difficult to avoid errors which in turn result in field errors. Thus, this manner of representing the errors, although it will not describe all possible perturbations, gives a sufficiently clear and simple idea of the arrangement of the magnet units, their identity, etc. Further adjustments of the magnetic field will have already the aim of suppressing particular perturbing harmonics.

Let us consider first a non-resonant deformation of the closed orbit $(x, z)_0$, the square of which, according to (2.11), is equal to

$$(x^2, z^2)_0 = \frac{e^2}{4p^2c^2K_0^4}\left\{\frac{|f|^2}{4\sin^2\pi\nu}\int_{\vartheta}^{\vartheta+2\pi}\int b(\vartheta')b(\vartheta'')f^*(\vartheta')f(\vartheta'')\times\right.$$

$$\times e^{i\nu(\vartheta''-\vartheta')}d\vartheta''d\vartheta' - \frac{f^2e^{2i\nu\vartheta}}{[1-e^{-2\pi i\nu}]^2}\int_{\vartheta}^{\vartheta+2\pi}\int b(\vartheta')b(\vartheta'')\times$$

$$\left.\times f^*(\vartheta')f^*(\vartheta'')e^{-i\nu(\vartheta'+\vartheta'')}d\vartheta'd\vartheta'' + \text{c.c.}\right\}. \qquad (3.16)$$

We shall now compute the probable value of this expression, assuming that there are M independent sectors with a length $2\pi/M$, which is much smaller than the wavelength of betatron oscillations. If the points ϑ', ϑ'' lie in different sectors, the averaged probable value $\overline{b(\vartheta')b(\vartheta'')} = 0$. If however they lie in the same sector, one may, since the sector is small, put down $\vartheta' = \vartheta''$ in the expression under the integral sign without making an appreciable error**. As a result we get

* Mathematically this means that the probable values of the amplitudes of the k first perturbation harmonics are practically the same so long $k \ll M$.

** The formulae of this paragraph do not pretend to be particularly accurate. We attempt to give an idea of the order of magnitude of the tolerances involved. A more elaborate treatment of the subject can be found in [4, 6, 9].

$$\overline{(x^2, z^2)_0} = \frac{e^2 \overline{b^2} 2\pi}{4p^2 c^2 K_0^4 M} \left\{ \frac{|f|^2}{4 \sin^2 \pi\nu} \int_{\vartheta}^{\vartheta+2\pi} |f|^2 \, d\vartheta' - \right.$$

$$\left. - \frac{f^2 e^{2i\nu\vartheta}}{[1-e^{-2i\pi\nu}]^2} \int_{\vartheta}^{\vartheta+2\pi} f^{*2} e^{-2i\nu\vartheta'} d\vartheta' + \text{c.c.} \right\}. \quad (3.17)$$

The second term in the brackets is always smaller than the first one, or, in the limit, equal to it. Besides, if we substitute for $|f|^2$ its maximum value $F^2(\mu)/\nu$ (see ch. 2, formula 4.15) we get finally

$$\sqrt{\overline{x_0^2, z_0^2}} < \frac{e\pi F^2(\mu) \sqrt{\overline{b_{z,x}^2}}}{pcK_0^2 M^{\frac{1}{2}} \nu |\sin \pi\nu|}. \quad 3.18$$

As an example of the use of this formula let us consider a very simple strong focusing accelerator made of $2N$ magnet blocks in each of which the field is constant along the azimuth and either increases or decreases along the radius with a field index n. If the blocks are displaced radially by a distance Δx, there appears on the ideal orbit a perturbation b_z given by (3.12). For the orbit excursion we get the following estimate

$$\sqrt{\overline{x_0^2}} < \frac{K^2 \pi F^2 |n|}{K_0^2 \nu N^{\frac{1}{2}}} \sqrt{\overline{(\Delta x)^2}}, \quad 3.19$$

where we put $|\sin \pi\nu| = 2^{-\frac{1}{2}}$, as the working point is usually chosen halfway between half-integer resonances:

$$\nu = k + \tfrac{1}{4}.$$

A somewhat similar formula links the vertical orbit excursion to the deformation of the median plane.

It is not difficult to see that the mean square orbit displacement will be much larger than the mean square error in the positioning of the magnet blocks. In large machines the factor is ten or twenty. Thus a misalignment of the magnets of ± 0.5 mm causes a mean square orbit deviation of 0.5 to 1 cm.

In the course of the design of an accelerator one would like to estimate not only the mean square orbit excursion, but also the degree of proba-

bility of still larger excursions. One can show [9] that the higher moments of the distribution satisfy the relation

$$\overline{x_0^{2k}} = k!(\overline{x_0^2})^k , \tag{3.20}$$

if $k \ll M$. This means that the probability of finding an excursion x_0 is given by a Gauss–Rayleigh distribution.

$$w(x_0) = \frac{2x_0}{\overline{x_0^2}} \exp\left(-\frac{x_0^2}{\overline{x_0^2}}\right), \tag{3.21}$$

Thus, in the example given above, x_0 can exceed the value obtained with a probability of 37 %, and twice this value with a probability of only 2 %.

If we consider machines with the same periodic structure, but with different values of n and N, we may observe that the phase gain of the betatron oscillation per period is μ and consequently that the stability depends only on the ratio n/N^2. Let the parameters of the accelerator vary in such a way that μ remains constant, as well as the quantity $|\sin \pi\nu|$, the optimum value of which is $2^{-\frac{1}{2}}$. Then ν in formula (3.19) will vary proportionally to N, and n proportionally to N^2 whereas $F(\mu)$ can be considered constant, as it depends only on the positions of the working point in the stability diagram. Thus the coefficient of multiplication grows as $N^{\frac{1}{2}}$ or $n^{\frac{1}{4}}$, i.e. inversely with the envelope of the betatron oscillations, which is proportional to $N^{-\frac{1}{2}}$. This circumstance sets a substantial limit to the focusing which one can achieve in practice, as any attempt to reduce the betatron oscillation amplitude by increasing N and n leads to an increase of the closed orbit distortion due to magnet imperfections. It is necessary to choose some compromise when designing a practical machine.

Besides closed orbit excursions one should, in a practical machine, exclude increases of the betatron oscillation amplitude due to parametric or coupling resonances. Let us look at these requirements from the statistical point of view.

We assume again that the magnet system may be divided into M sectors, in each of which there is a perturbation Δg. The mean probable value of these should be zero, $\overline{\Delta g} = 0$, but the perturbations in each sector should be statistically independent. Thus, from (2.7) we obtain

an expression for the mean square shift of the betatron oscillation frequency

$$\overline{(\Delta\nu)^2} = \frac{1}{4(2\pi)^2} \int_0^{2\pi}\int \overline{\Delta g(\vartheta')\Delta g(\vartheta'')}\, |f(\vartheta')|^2\, |f(\vartheta'')|^2\, \mathrm{d}\vartheta'\, \mathrm{d}\vartheta'' . \quad (3.22)$$

As previously, we may replace the quantity $|f|^2$ by its maximum value F^2/ν and obtain the upper limit for the frequency shift

$$\overline{(\Delta\nu)^2} < \frac{F^4(\mu)}{4M\nu^2}\, \overline{(\Delta g)^2} . \quad (3.23)$$

In the course of acceleration, the quantity $\overline{(\Delta g)^2}$ may vary, causing the working point to move about in the stability diagram. It is natural to require that the magnitude of this motion should remain say A times smaller than the cell dimension. Thus

$$\sqrt{\overline{(\Delta g)^2}} < \frac{\sqrt{M\nu}}{AF^2} . \quad 3.24$$

To illustrate this, let us take the parameters of a large strong focusing proton synchrotron

$$\nu = 8.75 , \qquad |n| = 360 , \qquad F \approx 1.5 , \qquad M = 2N = 240 , \qquad A = 10 .$$

The relative accuracy with which the field index $|n|$ must be maintained is then of the order of 1%. Usually supplementary tolerance requirements of the same order arise from the necessity of keeping the stop bands narrow due to parametric resonances. In fact this last requirement may be expressed mathematically in the form

$$|P| \ll \tfrac{1}{2} , \quad (3.25)$$

so that, if we estimate $\overline{|P|^2}$ on the basis of statistical considerations, we obtain formula (3.24) again.

One may estimate in a somewhat similar manner the tolerance requirements arising from the coupling resonances. The final choice of tolerances demands, as we have said, a computation of all the conditions of the practical case.

When one has fixed the tolerances on the field characteristics one

can compute the accuracy required for the setting of the individual magnet units. Use is made of (3.12) and (3.13), which relate the field perturbations to the displacements of the units from their ideal positions. In large machines ($n = 200$–400), the permissible asymmetry of field and field index $|n|$ is usually a fraction of one per cent, corresponding to positioning errors of less than 1 mm.

The parameters of the various machines indicated in chapter 7 give an idea of the order of magnitude of the tolerances with which one has to deal in practice.

The considerations on tolerances in this paragraph have been based on the linear theory of betatron oscillations. The computation of non-linear effects which is carried out in the following paragraphs may in certain cases somewhat alter the value of these tolerances (see for instance [13, 14].

4. BASIC NON-LINEAR EFFECTS

The results of the previous paragraphs were obtained on the assumption that the particle motion was described by linear equations in the neighbourhood of the ideal orbit. In present accelerators this assumption is quite well realized. However, it is sometimes necessary to take into account some non-linear corrections. This is evident if one considers the behaviour of the closed orbit on an integer resonance line. The linear theory leads to an infinitely large excursion from the ideal orbit. This is evidently incorrect, as the existence of the closed orbit in any arbitrary field does not depend on the parameters of betatron oscillations, and consequently on the presence or absence of resonances. This infers that even at resonance the orbit distortion may remain finite. The reason for this lies in the non-linear character of the full equations of motion. In fact the linearization of the original equations, that is the neglecting of terms of higher order in x and z, is only permissible when these quantities are small. And this is not always the case in the neighbourhood of resonance, where the linearized theory gives an infinite growth of oscillation amplitude.

There are two reasons for the appearance of non-linear terms in the equations of motion. Firstly, the magnetic field is a non-linear function of the co-ordinates, for instance, of the radius. Non-linearities of this

type can, in principle, be suppressed by means of various correcting lenses or coils. Secondly, even in a purely linear field, the equations of motion are non-linear in any ideal or real magnet system. So the attribution of non-linear effects to the defects of the magnetic field is somewhat artificial. But, with ideal systems, it is nearly always possible to neglect the non-linearities, assuming that the oscillations are sufficiently small. In real systems with field errors, especially in the neighbourhood of the 'stop bands', the action of the non-linearity of the oscillation may be quite important.

Let us consider qualitatively the phenomena to which the non-linearity can lead. Probably the most important peculiarity of the anharmonic oscillator is the dependence of its oscillation frequency upon the amplitude. It is clear that this dependence must, in general, limit the amplitude growth in a linear resonance. As long as the amplitude increases, the eigenfrequency changes and drifts away from the resonance value. This phenomenon can play a useful role in accelerators. It has been proposed to remove the most important resonances by introducing artificially a strong non-linearity. However, the application of this idea presents fundamental difficulties as new resonances appear which are different from those described above*. In fact, if the oscillations are not coupled, their equations have the form

$$\begin{aligned} x'' + g_x x &= F_x(x, z, x', z', \vartheta)\,, \\ z'' + g_x z &= F_z(x, z, x', z', \vartheta)\,, \end{aligned} \tag{4.1}$$

where F_x and F_z account for the differences between the real system and the ideal one, as well as for small non-linear effects. Let us assume that the motion is described to a first approximation by the expressions (4.5) of chapter 2, and let us introduce them into (4.1), which we can expand in powers of x and z. It is then easily seen that the perturbing forces F_x and F_z contain in general all possible frequencies which can be obtained by linear combinations of ν_x, ν_z and 1. Already with this simple representation it is natural to expect resonance phenomena from non-linearities when the following condition is fulfilled

* Moreover, in systems where the parameters vary slowly, as is always the case in practice, the limitation of resonances by non-linearities is far less effective than one would expect from our simple explanation.

$$k_x v_x + k_z v_z + q \approx 0 \quad (k_{x,z} \text{ and } q \text{ are integers}), \tag{4.2}$$

then even small coupling terms between the different degrees of freedom may cause an intense exchange of energy between them. The exact theory shows [10, 15, 16] that (4.2) is, in fact, the general condition for the excitation of resonances in a system with three degrees of freedom when the non-linearity is weak.

It is important to stress that when $q = 0$ we have only to deal with a coupling of transverse oscillations which cannot lead to an infinite increase of the amplitude, as has already been shown. As k_x and k_z must have different signs in this case, this coupling resonance is called a difference resonance, analogous to (2.47). When $q \neq 0$, however, the transverse oscillations are coupled with the longitudinal motion, which has an infinite reserve of energy, and they can reach any amplitude.

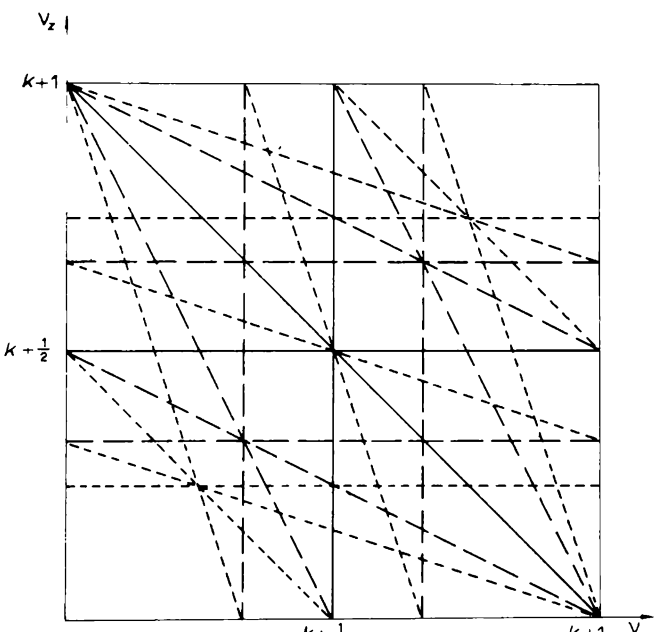

Fig. 28. Resonance lines of various orders: ——— linear resonances, – – – cubic resonances, ––––– fourth order resonances.

As we shall see later, the accurate theory actually indicates the possibility of betatron resonances when (4.2) is fulfilled. We shall call $|k_x| + |k_z|$ the order of the resonance. Thus the linear resonances considered above are of the first or second order. Resonances of higher order can appear only in the presence of non-linearities.

At first glance one might think that it is fundamentally impossible in a real system to find a working point far from a resonance, as for large enough k_x, k_z and q, there will always be a line close to any given point in the stability diagram. However, we should note that the 'strength' of the resonance, that is the width of the resonance band and its danger under actual conditions, decrease rapidly with increasing order. Thus a resonance of the fourth order will only create difficulties in quite special cases.

On the other hand, the non-linear resonances divide the already small stability cell in still smaller 'sub-cells'. This situation is demonstrated in fig. 28, where resonances up to the fourth order inclusive are represented in the (ν_x, ν_z) plane, as well as the difference coupling resonances.

It is interesting to observe that in contrast to linear resonances, which can only be excited by low order harmonics of the magnetic field and its gradient, non-linear resonances take place even in an ideal magnetic field. In fact, as we will see later, a resonance of the kind (4.2) will be excited by the qth harmonic of the azimuthal dependence of the field and its derivatives. Within the limits of the first stability region we have

$$0 < \nu_{x,z} < \tfrac{1}{2}N$$

so that, in the cases of first and second order resonances, condition (4.2) can only be fulfilled by harmonics of order $q < N$, which are absent in an ideal field. The situation is different for non-linear resonances, which may be excited by harmonics whose order is a multiple of N. So a cubic one-dimensional resonance is excited in an ideal field when $\nu \approx \frac{1}{3}N$ by the harmonic $q = N$, and so on. As the periodicity is, in general, quite marked in the ideal field, harmonics of this order are evidently very relevant and the danger of such 'ideal' resonances is increased. However, it is easily seen that the 'ideal resonance' lines are much rarer in the stability diagram than lines of perturbation resonances (see fig. 28).

The influence of non-linearities is not limited to these effects. For example, they can cause betatron oscillations of sufficiently large ampli-

tude to be unstable far away from resonances, they can influence the choice of tolerances, especially for strong focusing machines, etc. The discussion of all non-linear effects clearly lies outside the framework of this book, the more so as most of them are still insufficiently investigated. So we shall limit ourselves to examining in detail the influence of non-linearities on betatron oscillation resonances, this part of the question being at least for the present, the most essential. Let us stress again that in most cases of practical interest, betatron oscillations are nearly linear, non-linear effects calling for only small corrections. This assumption forms the basis of the majority of papers devoted to the question (see [6] and [13–19]).

For simplicity's sake and to avoid cumbersome computations, we shall assume below that the ideal system has a plane of symmetry in which the ideal orbits must lie. Also we shall consider only conditions under which the motion is one-dimensional. The calculations developed in this way have been generalized to include the case of coupled oscillations [18], but their review would occupy too much space and we shall restrict ourselves to a brief account of the question.

5. GENERAL THEORY OF NON-LINEAR RESONANCE. ONE-DIMENSIONAL OSCILLATIONS

Let us look now in more detail at the general phenomena which accompany the action of small non-linearities on a system oscillating linearly. This part of oscillation theory is now well studied and presented in a series of monographs [10, 20, 21]. We shall dwell only on questions which are of immediate interest for particle dynamics in accelerators, in particular, resonance phenomena.

One may study resonances of any order by a single method [6, 13–19] of which we have already made practical use in a particular case when considering linear resonances. The basic idea is that the integrals of the linear equations of motion cannot remain constant in the presence of perturbations and non-linearities, but that their rate of change in most physically interesting cases is small. So one can use the averaging method and obtain simplified equations which can be either integrated directly or used to follow the motion in phase space. It is convenient to use variables which lead to simplified equations in canonical form. In

most cases this simplifies the calculation of the phase trajectories.

One describes linear oscillations around the ideal orbit by Hamilton's quadratic form (ch. 2, 5.9), the coefficients of which depend on ϑ with a period ϑ_0. To describe the influence of magnet defects and of non-linearities we write the Hamiltonian of the real system in the form

$$\mathscr{H}(x, z, p_x, p_z, \vartheta) = \mathscr{H}_0(x, z, p_x, p_z, \vartheta) + \mathscr{H}_1(x, z, p_x, p_z, \vartheta), \qquad (5.1)$$

where $\mathscr{H}_0$ is the quadratic form (5.9) of chapter 2, relative to the ideal system, and the small term $\mathscr{H}_1$ represents field errors and non-linearities.

As canonical variables it is convenient to take here the action and the phase (I, η), which we have in fact used already in earlier paragraphs. The integral of action I depends very simply on the oscillation amplitude, being proportional to its square. Thus a survey of phase trajectories (I, η) gives an immediate answer to the question of the amplitude limits near resonance.

One can consider the transition to (I, η) formally as a canonic transformation defined by a generating function of the type (6.9) of chapter 2. We may write down at once transformation formulae practically equivalent to (2.4)

$$\begin{aligned}(x, z) &= \left[\frac{2I_{x,z}}{K_0 p}\right]^{\frac{1}{2}} (f_{x,z}\, e^{i\nu_{x,z}\vartheta + i\eta_{x,z}} + \text{c.c.}), \\ p_{x,z} &= [2I_{x,z} K_0 p]^{\frac{1}{2}} [(f' + i\nu f)_{x,z}\, e^{i\nu_{x,z}\vartheta + i\eta_{x,z}} + \text{c.c.}],\end{aligned} \qquad (5.2)$$

where the frequencies ν_x, ν_z and Floquet functions $f_{x,z}$ apply to the ideal system. It is easy to check that the 'ideal' part of the Hamiltonian is a constant in the new variables, which is evident, as I and η in the absence of perturbations are integrals of the linear equations of motion. Thus the canonic equations of motion have the form

$$I' = -\frac{\partial \mathscr{H}_1}{\partial \eta}, \qquad \eta' = \frac{\partial \mathscr{H}_1}{\partial I}, \qquad (5.3)$$

where the Hamiltonian $\mathscr{H}_1$ must of course be expressed in the variables I and η.

Let us note now that the canonic equations (5.3) have the standard form for the averaging method, as the derivatives I' and η' are small since the Hamiltonian $\mathscr{H}_1$ accounting for the perturbations is small.

Thus the simplified equations in the first approximation, which is usually sufficient*, can be obtained from (5.3) by simply averaging over the explicit variable ϑ, and keep the canonical form with the Hamiltonian $\langle \mathscr{H}_1 \rangle$**.

For clarity we shall first examine in detail the case of one-dimensional oscillations (for instance, in the x-direction). In this case one can represent $\mathscr{H}_1$ as a power series

$$\mathscr{H}_1 = \sum_{\kappa,\lambda=0}^{\infty} h_{\kappa,\lambda}(\vartheta) x^{\kappa} p_x^{\lambda} = \sum_{q=-\infty}^{+\infty} \sum_{\kappa,\lambda=0}^{\infty} h_{\kappa,\lambda}^{(q)} x^{\kappa} p_x^{\lambda} e^{iq\vartheta}, \tag{5.4}$$

where we have expanded the coefficients $h_{\kappa,\lambda}(\vartheta)$ in a Fourier series. Introducing here (5.2) and grouping together terms having the same exponent, we obtain

$$\mathscr{H}_1 = \sum_{q=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} e^{i(q+k\nu)\vartheta + i\eta k} \sum_{m=|k|}^{\infty} u_{q,k,m}(\vartheta) I^{\frac{1}{2}m}. \tag{5.5}$$

Here the coefficients $u_{q,k,m}(\vartheta)$, having a period ϑ, are expressed by the Floquet functions of the ideal system and, in addition, are proportional to the qth harmonic of the perturbation. The sum in formula (5.5) applies only to even m, if q is even, and uneven m if q is uneven. Expressions for some coefficients $u_{q,k,m}$ corresponding to the more important resonances will be found later (in § 7).

In (5.5) we have considered non-linearities of any order which is not at all necessary in practice. In fact, if non-linearities are mere corrections and if betatron oscillations are sufficiently small, the series (5.5) of powers of $I^{\frac{1}{2}}$ converges rapidly. Thus it will be sufficient to keep the first two or three terms which will describe fairly well the behaviour of the amplitude around resonance.

As far as devices with large non-linearities are concerned, one has to make further investigations based, as a rule, on numerical methods.

The structure of Hamiltonian (5.5) is rather simple and permits a

* We should just mention the following circumstance. At a large distance from resonance the simplified equations have only the trivial solution $I = \text{const}$. If it is required to study the non-resonant behaviour of orbits, one should use an improved first approximation (appendix C.3) considering small rapidly oscillating terms. For details see [10].

** Passage to the simplified equation can be considered also as a canonical transformation. In this connection, see for instance [15, 17].

series of conclusions. The presence of the factor $\exp(ikv\vartheta)$ shows that in general all terms under the triple summation sign are rapidly oscillating functions of ϑ with the exception of the term corresponding to $k=0$. Thus in the non-resonant case the averaged Hamiltonian has the form

$$\langle \mathscr{H}_1 \rangle = \sum_q [I\langle u_{q,0,2}\, e^{iq\vartheta}\rangle + I^2\langle u_{q,0,4}\, e^{iq\vartheta}\rangle] + O(I^3)\,. \tag{5.6}$$

As η does not appear in this expression, one sees that the Hamiltonian (5.6) describes simply the change of frequency in the real system with respect to the ideal one. In fact, it follows from (5.6) and (5.3) that

$$I = \text{const.}\,,$$

$$\eta' = \Delta\nu + 2I\sum_q \langle u_{q,0,4}\, e^{iq\vartheta}\rangle = \Delta\nu + 2\zeta I\,, \tag{5.7}$$

where $\Delta\nu$ is the frequency shift in the linear system, given by formula (2.7), as one can show that

$$\sum_q \langle u_{q,0,2}\, e^{iq\vartheta}\rangle \equiv \tfrac{1}{2}\langle |f_x|^2\, \Delta g_x\rangle\,. \tag{5.8}$$

As in the preceding paragraph, we shall not write down the 'linear' correction to the frequency, and we shall mean by ν the frequency of the linear oscillations in the real system. The expression (5.7) shows further that the frequency depends on the amplitude (second term in 5.7). This circumstance is of no great concern far away from resonance (I small) but, as we shall see below, it exerts a great influence on the behaviour of phase trajectories close to resonance. The higher-order-terms which we have neglected are not significant under real conditions on account of the considerations referred to above.

The amplitude dependence of the frequency is characterized by the quantity $\zeta = \Sigma_q(u_{q,0,4}\, e^{iq\vartheta})$. Depending on the sign of ζ, the frequency can grow ($\zeta > 0$) or decay ($\zeta < 0$) as the amplitude increases. (In oscillation theory, such systems are sometimes called respectively systems with a strong or weak characteristic restoring force.) We shall see below that it is just this parameter, which is basically bound to the cubic non-linearity in the magnetic field, that is responsible for the stabilizing action of the non-linearity on the resonance of betatron oscillations.

Let us consider the general condition for a resonance of the kth order

$$\nu + \frac{q}{k} = \delta \ll 1\,, \qquad k \text{ and } q \text{ integers} \tag{5.9}$$

of which the linear one-dimensional resonances examined above are particular cases. Averaging expression (5.5) we then obtain

$$\langle \mathscr{H}_1 \rangle = \zeta I^2 + I^{\frac{1}{2}k}\left[e^{ik\vartheta + i\eta k}\langle u_{q,k,k}\rangle + \text{c.c.}\right] + \dots\,, \tag{5.10}$$

where we have again dropped unessential terms, proportional to $I^{\frac{1}{2}k+1}$, $I^{\frac{1}{2}k+2}$ and so forth. Let us only mention that because of some of these terms, resonances of orders $2k, 3k$, etc., can be excited at the same time as the kth order resonance.

Let us begin the study of resonances of different orders in the presence of non-linearities with the case of the main resonance ($k = 1$). As shown in the previous paragraph, the behaviour of the closed orbit is in this case also strongly influenced by the parametric resonance. To compute the latter, one must keep in the right-hand side of (5.10) the term describing the parametric perturbation ($\sim I^{2k/2}$):

$$I\left[e^{2i\vartheta\delta + 2i\eta}\langle u_{2q,2,2}\rangle + \text{c.c.}\right]. \tag{5.11}$$

Higher-order terms in I may usually be neglected.

The main resonance has its main effect on the behaviour of the closed orbit, which is defined by the following conditions (see 2.28–2.30):

$$I = \text{const.} = I_0\,, \qquad \eta' = -\delta = \text{const.} = \eta_0'\,, \tag{5.12}$$

or

$$\frac{\partial}{\partial \eta}\langle \mathscr{H}_1 \rangle = 0\,, \qquad \frac{\partial}{\partial I}\langle \mathscr{H}_1 \rangle = -\delta\,.$$

These relations yield the following pair of equations for determining I_0, η_0:

$$\begin{aligned} 2I_0^{\frac{1}{2}}|\langle u_{2q,2,2}\rangle| \cos(2w_0 + \beta) + |\langle u_{q,1,1}\rangle| \cos w_0 = 0\,, \\ I_0^{\frac{1}{2}}(\delta - 2|\langle u_{2q,2,2}\rangle| \sin(2w_0 + \beta)) + 2\zeta I_0^{\frac{1}{2}} - |\langle u_{q,1,1}\rangle| \sin w_0 = 0\,, \end{aligned} \tag{5.13}$$

where

$$w_0 = \delta\vartheta + \eta_0 + \text{const.}\,,$$

and where β is a constant depending on the phase of the perturbation.

It is easy to see that for a given value of the perturbations $|\langle u_{2q,2,2}\rangle|$ and $|\langle u_{q,1,1}\rangle|$, the closed orbit excursion will be maximum for $\cos\beta = 0$. In fig. 29 we have drawn the resonance curve, that is the dependence of $|a_0|$ upon the distance from the resonant frequency. One sees clearly that the closed orbit excursion for $\zeta \neq 0$ remains finite for any value of δ.

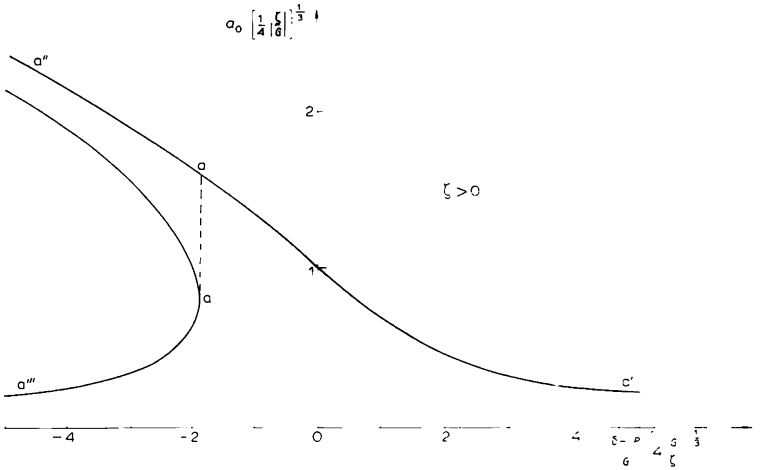

Fig. 29. Closed orbit amplitude in the neighbourhood of a main resonance in the presence of a stabilizing non-linearity. The symbols are the same as in § 2.

An interesting feature of the equation we have obtained is also that it has several solutions. In fact, for sufficiently large negative δ we find three closed orbits, two of which are stable, as one can show (in fig. 29 the solutions are indicated by thick lines). We should stress at once that this circumstance can lead to the 'lock-on' phenomenon for particles near resonance when the distance from resonance changes slowly. Let us suppose that we are passing through resonance in fig. 29 from left to right. Particles which were oscillating around the stable orbit with the smaller excursion will jump at point a to the orbit with the larger amplitude, and the motion will continue around the points of curve a–a' as equilibrium positions. The situation is somewhat different when passing through the resonance in the other direction. After arriving at point a, a part of the particles, depending on initial conditions and of

motion, will continue on curve aa'''; another part will travel on the other branch aa'', which leads to extremely large excursions of their orbits from the equilibrium orbit. Such an asymmetry in the crossing of a resonance can be explained by the fact that for positive ζ, the increase in amplitude when approaching resonance from the left accelerates the passage through it, whereas it slows it down when coming from the right, as the frequency tends to increase with the amplitude. For negative ζ the picture is analogous, but the 'lock-on' will take place when passing through the resonance from left to right.

Let us now go over to studying the phase trajectories in the neighbourhood of a resonance of any desired order ($k > 1$). We notice that the following simple change in the phase variable

$$w = \delta\vartheta + \eta \tag{5.14}$$

conserves the canonical form of the simplified equations, whereby the Hamiltonian (5.10) comes out independent of ϑ and is, therefore, an integral of the motion. Let us introduce the dimensionless quantities

$$\mathscr{I} = I/I_i\ ; \qquad \Delta = \frac{\delta}{2I_i^{\frac{1}{2}k-1}|\langle u_{q,k,k}\rangle|}\ ; \tag{5.15}$$

$$\xi = \frac{\zeta}{2} I_i^{2-\frac{1}{2}k}|\langle u_{q,k,k}\rangle|^{-1}\ ,$$

where as usual I_i indicates the initial value of the quantity I, and where Δ and ξ correspond respectively to the distance from resonance and to the initial non-linear frequency shift relative to the width of the stop band. The parameters Δ and ξ characterize intrinsically a resonance of any order and determine the phase trajectories, whose equations have the simple form

$$\mathscr{I}\Delta + \xi\mathscr{I}^2 + \mathscr{I}^{\frac{1}{2}k}\cos(kw+\beta) = \text{const.} \tag{5.16}$$

On figs. 30 and 31 the polar phase diagrams are given for resonances of order $k=2$ to $k=5$, taking into account the non-linear frequency shift ($\xi \neq 0$), and leaving this out ($\xi = 0$). One easily sees that in the first case, the phase trajectories are closed even inside the stop bands (for $|\Delta| < 1$). It is true that for resonances of higher order ($k > 4$), this is only correct for sufficiently small amplitudes and that the instability

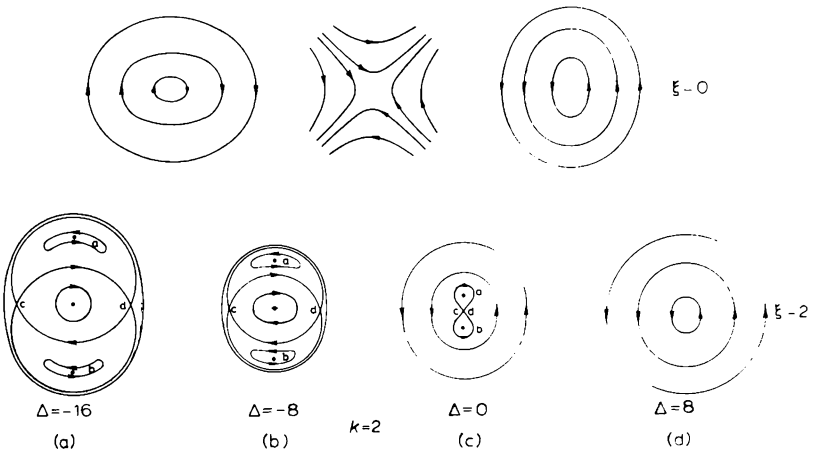

Fig. 30. Phase trajectories in the neighbourhood of a parametric resonance [17].

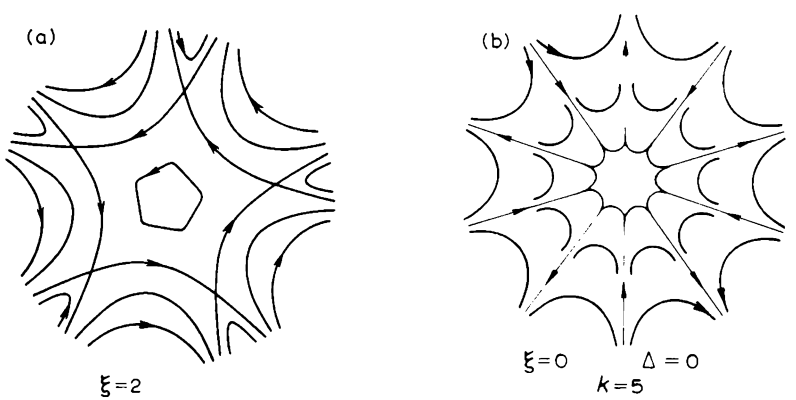

Fig. 31. Phase trajectories for a resonance of the fifth order [17].

remains for large amplitudes. One should remember, however, that the structure of the phase trajectories in this region must be influenced by the higher-order non-linear terms we have neglected.

Another interesting feature of the phase diagrams is the appearance of additional special points, where the motion is stable or unstable (for instance points a, b, c, d in fig. 30). Each of these points corresponds to

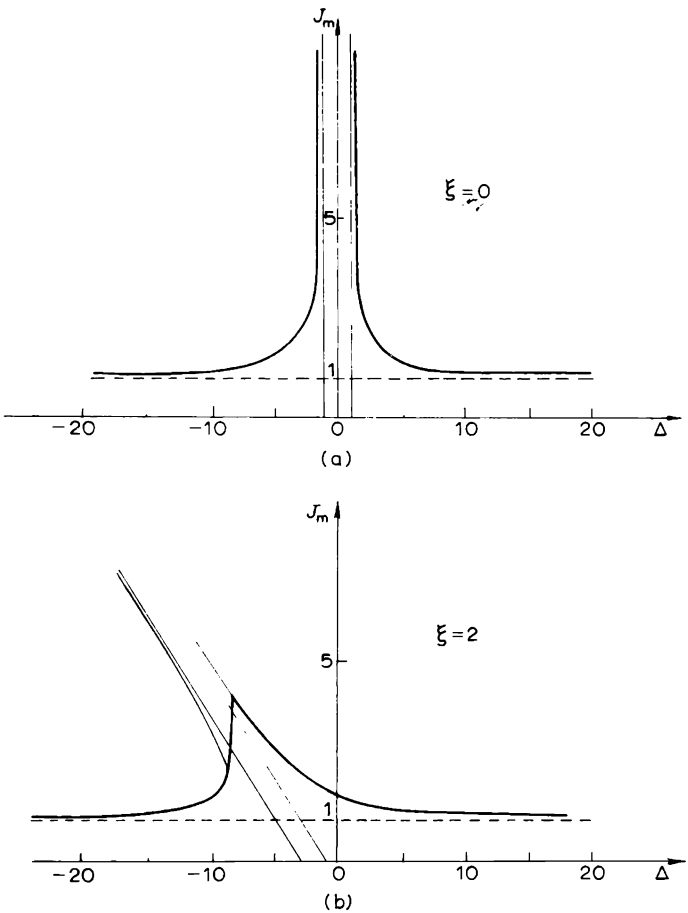

Fig. 32. Resonance curve for a parametric resonance [17].

motion with constant I and w, and therefore indicates the appearance of new closed orbits, which differ however from the original one in the fact that they do not close up after one, but after k revolutions.

For operation of the accelerator, the most interesting feature of the motion is the maximum amplitude $\mathcal{J}_m$ which can be reached by the representative point in the course of its motion along the phase trajec-

tory, in other words, the beating ratio which corresponds to the worst initial conditions. The dependence of $\mathscr{I}_m$ upon the distance Δ from resonance is shown in fig. 32 for the case $k=2$, that is for the half-integral (parametric) resonance. On the upper graph, which corresponds to $\xi=0$, that is to motion without non-linear phase drift, the curve goes to infinity asymptotically when Δ approaches $+1$ on the right and -1 on the left. Inside the interval $-1<\Delta<+1$ the beating ratio is not limited, which corresponds, as one will easily see, to a 'forbidden' region in the linear theory of resonances. The width of this stop band is given by (2.32) and by virtue of (5.15) does not depend on the initial amplitude. For $\xi\neq 0$, the curves are bent on one side or the other depending upon the sign of ξ, and this leads to the disappearance of the forbidden band* and to the limitation of the possible beating ratio, whose maximum value is given by the full curve.

The graphs for resonances of higher order have the same appearance. For $\xi=0$ again we find forbidden bands, whose widths are determined by the points of intersection of the curves $\mathscr{I}_m(\Delta)$ with the line $\mathscr{I}_m=1$. However, as follows from (5.15), the width of the band depends for $k>2$ on the initial amplitude.

$$\delta_{\text{res.}} \sim I_i^{\frac{1}{2}k-1} . \tag{5.17}$$

When there is a non-linear frequency drift ($\xi\neq 0$) the forbidden band always vanishes in the case $k=3$. For resonances of higher order this happens only for sufficiently large ξ. As, according to (5.15), $\xi \sim I_i^{2-\frac{1}{2}k}$, ξ will increase with increasing initial amplitude for $k<4$, decrease for $k>4$, and be independent of I_i for $k=4$. This means that for any value $\xi\neq 0$, resonances with $k>4$ do not perturb the stability of oscillations of sufficiently small amplitude. For $k=4$ the stability condition is $|\xi|\geqslant 1$.

To estimate how dangerous will be beats which occur near resonance in the presence of a non-linear frequency drift, it is convenient to use the graph on fig. 33, where the maximum possible value of $\mathscr{I}_m$ (for any given distance from resonance) is plotted as a function of the stabilizing parameter $|\xi|$. From fig. 33 we see that if we want $\mathscr{I}_m$ not to exceed 2, i.e. the initial amplitude not to increase by more than a factor 1.5, values

* Of course, it is the region where the amplitude grows to infinity which vanishes. The question of tolerable beats will be examined later.

of $|\xi|$ of the order of 10 to 20 are required. Let us stress again here that these considerations only apply to 'static' conditions under which the systems parameters do not vary with time. In the case of dynamic systems with parameters varying slowly with time, the results can change seriously.

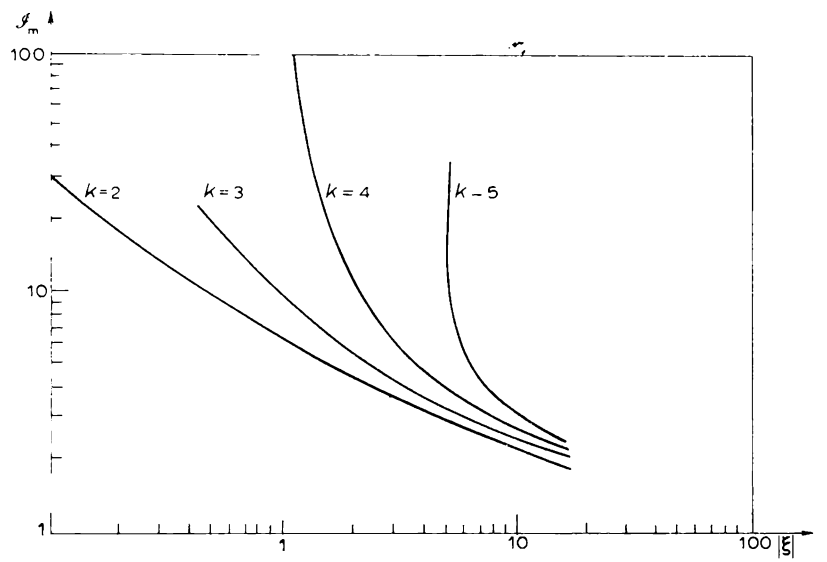

Fig. 33. Dependence of the beating ratio on the stabilizing non-linearity [17].

One may also gain a clear view of the action of the non-linear frequency shift upon the oscillation amplitude by looking at the picture in the (ξ, Δ) plane. According to (5.16), the lines of equal maximum amplitude in this plane are the straight lines

$$\Delta = -\xi(1+\mathcal{J}_m) \mp \frac{1+\mathcal{J}_m^{\frac{1}{2}k}}{\mathcal{J}_m - 1}, \tag{5.18}$$

where the upper or lower sign has to be taken, depending whether the quantity Δ lies on the left or right of the break point of the curve in fig. 32.

Such a diagram has been constructed in fig. 34 for the case of the

parametric resonance ($k = 2$). The boundary at which one has to change the sign in formula (5.18) is the envelope of the family of straight lines given by (5.18). If the maximum permissible value of the beating is given, it is possible to determine from the graph in fig. 34 the forbidden region for the values of the parameters Δ and ξ. (For instance, if $\mathscr{I}_m = 3$ this region is limited by the curve *abcd* on fig. 34.) To avoid confusion let us stress that the graphs on figs. 33 and 34 should be used with caution

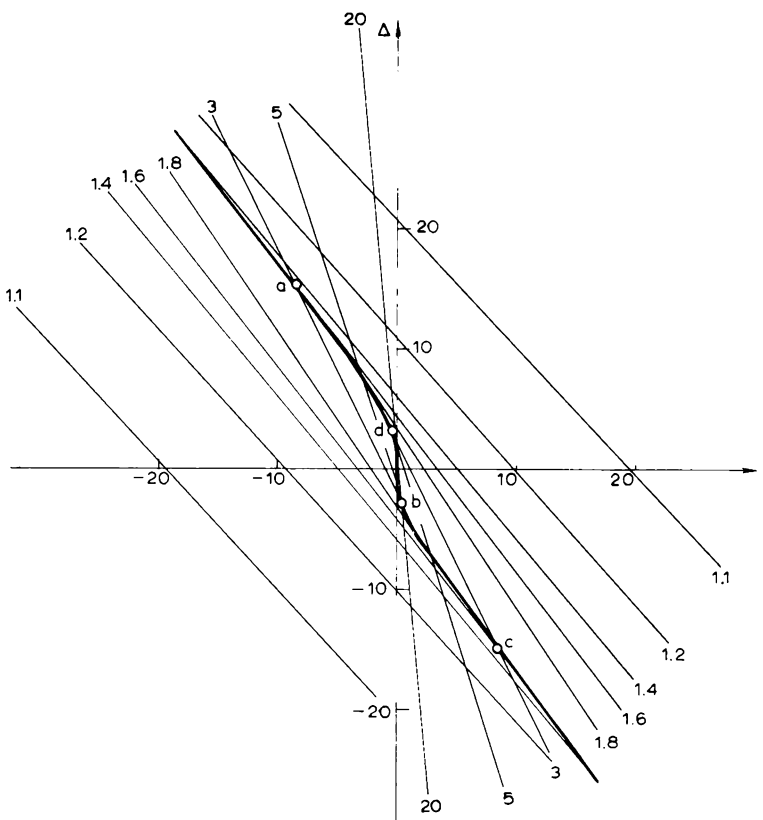

Fig. 34. Lines of equal maximum amplitudes for the case of the parametric resonance. Along the abcissa the quantity 2ξ has been indicated.

even when the system parameters vary slowly, as the amplitude does not depend in this case only on the instantaneous values of ξ and Δ but also on the previous history of the process.

6. NON-LINEAR RESONANCES OF TWO-DIMENSIONAL OSCILLATIONS

One can generalize the scheme outlined in the preceding section for the case of coupled x- and z-oscillations, that is consider the influence of the non-linearity on various coupling resonances [18]. It should be noted however that in doing this the intermediate calculations will be rather complicated so we shall omit them wherever possible.

The Hamiltonian (5.4) in the case that we consider must be expanded in powers of x, z, p_x, p_z, and after going over to the variables action-phase, it has the following form

$$\mathscr{H}_1 = \sum_q \sum_{k_x, k_z} \exp i[(q + k_x \nu_x + k_z \nu_z)\vartheta + \eta_x k_x + \eta_z k_z] \times \times \sum_{\substack{m_x = |k_x| \\ m_z = |k_z|}}^{\infty} u_{q, k_x, k_z, m_x, m_z} I_x^{\frac{1}{2}m_x} I_z^{\frac{1}{2}m_z}, \qquad (6.1)$$

where the coefficients $u(\vartheta)$ are various combinations of the Floquet functions in the two directions of oscillation.

The structure of the Hamiltonian remains the same as in the one-dimensional case. It consists of slowly varying terms describing the averaged motion, and rapidly oscillating ones which give corrections only in the higher approximations of the averaging method.

The only terms which do not average zero in the resonance case are those describing the variation of frequency ($k_x = k_z = 0$):

$$\langle \mathscr{H} \rangle = \zeta_x I_x^2 + \zeta_z I_z^2 + \zeta_{xz} I_x I_z + \Delta\nu_x I_x + \Delta\nu_z I_z + \ldots, \qquad (6.2)$$

when

$$\zeta_x = \left\langle \sum_q e^{iq\vartheta} u_{q,0,0,4,0} \right\rangle, \qquad \zeta_z = \left\langle \sum_q e^{iq\vartheta} u_{q,0,0,0,4} \right\rangle,$$

$$\zeta_{xz} = \left\langle \sum_q e^{iq\vartheta} u_{q,0,0,2,2} \right\rangle. \qquad (6.3)$$

As in the preceding section, we find in addition to the linear frequency shift a non-linear one, which depends on the oscillation amplitude. It is important to keep in mind that the frequency shift in each direction of oscillation depends, in general, on the two amplitudes I_x and I_z, which are the respective integrals of motion as in the one-dimensional case, far from resonance. Also one should note that in some cases which are important in practice, $\zeta_{xz} \approx 0$, so that the frequency in each direction of oscillation may well depend on one amplitude only.

More interesting is the resonance case for which the following condition is fulfilled in the actual system

$$q + k_x \nu_x + k_z \nu_z = \delta(k_x^2 + k_z^2)^{\frac{1}{2}} \ll 1. \qquad (6.4)$$

We note that the parameter δ, defined by eq. (6.4) has, as in the one-dimensional case, a simple geometric signification, namely the distance of the working point from the resonance line under consideration.

One can put the averaged Hamiltonian into the form

$$\langle \mathscr{H}_1 \rangle \approx \frac{(k_x^2+k_z^2)^{\frac{1}{2}}}{k_x}\,\delta I_x + \frac{(k_x^2+k_z^2)^{\frac{1}{2}}}{k_z}\,\delta I_z + \zeta_x I_x^2 + \zeta_z I_z^2 + \zeta_{xz} I_x I_z + \\ + 2|\langle u\rangle| \cos(k_x w_x + k_z w_z + \beta) I_x^{\frac{1}{2}|k_x|} I_z^{\frac{1}{2}|k_z|}, \tag{6.5}$$

$$u \equiv u_{q,k_x,k_z|k_x|,|k_z|},$$

where, as in the preceding section, we have gone over to the new phase variables.

$$w_x = \eta_x + \delta\vartheta \left(1 + \frac{k_z^2}{k_x^2}\right)^{\frac{1}{2}}, \qquad w_z = \eta_z + \delta\vartheta \left(1 + \frac{k_x^2}{k_z^2}\right)^{\frac{1}{2}}.$$

In the case of one-dimensional oscillations it was enough to know the equations of the phase trajectories to determine the limits of the amplitude variation. Now, however, it is indispensable, for the same scope, to have one relation more, which should relate I_x and I_z, that is to find another integral of the motion. Let us show that this is possible, if we make use of a few symmetry properties of the equations of motion.

In fact, from expression (6.5) of the Hamiltonian one sees easily that

$$\frac{1}{k_x}\frac{\partial}{\partial w_x}\langle \mathscr{H}_1 \rangle = \frac{1}{k_z}\frac{\partial}{\partial w_z}\langle \mathscr{H}_1 \rangle, \tag{6.6}$$

so that we obtain with the canonical equations of motion,

$$\frac{I_x'}{k_x} - \frac{I_z'}{k_z} = 0$$

and consequently

$$k_z I_x - k_x I_z = \text{const.} \tag{6.7}$$

From this follows an extremely important and general theorem.

If k_x and k_z have opposite signs (if we have a difference-resonance) the amplitudes in both directions remain limited in the course of the motion, and the resonance does not lead to instability. An increase of the amplitude in one direction in this case leads to a decrease in the other, so that an exchange of energy takes place between the different degrees of freedom. If, however, k_x and k_z have the same sign (sum-resonance), an increase or decrease of amplitude takes place at the same time for the x- and z-oscillations. This result is clearly the generalization of the rule of the same kind we had obtained in § 2 for the linear coupling resonance.

If we make use of the two integrals of motion (6.5) and (6.7), we may determine the limits of amplitude variations in the neighbourhood of coupling resonances of any arbi-

trary order. For this it is convenient to go over again to the dimensionless variables $\mathscr{I}_{x,z}$ (see eq. 5.15) with which eqs. (6.5) and (6.7) take the general form

$$\Delta_x \mathscr{I}_x + \Delta_z \mathscr{I}_z + \xi_x \mathscr{I}_x^2 + \xi_z \mathscr{I}_z^2 + \xi_{xz} \mathscr{I}_x \mathscr{I}_z + \\ + \mathscr{I}_x^{\frac{1}{2}|k_x|} \mathscr{I}_z^{\frac{1}{2}|k_z|} \cos(k_x w_x + k_z w_z + \beta) = C = \text{const.}, \quad (6.8)$$

$$k_z \mathscr{I}_x - k_x \mathscr{I}_z = k_z - k_x = \text{const.},$$

where

$$\Delta_{x,z} = \frac{\delta}{2} \frac{(k_x^2 + k_z^2)^{\frac{1}{2}}}{k_{x,z}} I_{ix,z} I_{ix}^{-\frac{1}{2}|k_x|} I_{iz}^{-\frac{1}{2}|k_z|} |\langle u \rangle|^{-1},$$

$$\xi_{x,z} = \tfrac{1}{2} \zeta_{x,z} I_{ix,z}^2 I_{ix}^{-\frac{1}{2}|k_x|} I_{iz}^{-\frac{1}{2}|k_z|} |\langle u \rangle|^{-1}, \quad (6.9)$$

$$\xi_{xz} = \tfrac{1}{2} \zeta_{xz} I_{ix}^{1-\frac{1}{2}|k_x|} I_{iz}^{1-\frac{1}{2}|k_z|} |\langle u \rangle|^{-1}.$$

Constant C in formula (6.8) may take any value in the range

$$-1 < C - \Delta_x - \Delta_z - \xi_x - \xi_z - \xi_{xz} < 1. \quad (6.10)$$

With (6.8) one can study the phase trajectories, the dependence of the beating ratio on the distance from resonance etc., as was done in the one-dimensional case. Let us examine here only a few general aspects of the situation deriving from eq. (6.8). First of all, the motion can, evidently, only take place in regions where

$$\mathscr{I}_x^{\frac{1}{2}|k_x|} \mathscr{I}_z^{\frac{1}{2}|k_z|} \geqslant |\Delta_x \mathscr{I}_x + \Delta_z \mathscr{I}_z + \xi_x \mathscr{I}_x^2 + \xi_z \mathscr{I}_z^2 + \xi_{xz} \mathscr{I}_x \mathscr{I}_z - C|. \quad (6.11)$$

At the same time, with the second relationship (6.8) one finds that for large $\mathscr{I}$ the right-hand side of inequality (6.11) grows as $\mathscr{I}^2$ whereas the left-hand side as $\mathscr{I}^{\frac{1}{2}(|k_x|+|k_z|)}$ (by $\mathscr{I}$ one may understand any of the quantities $\mathscr{I}_{x,z}$). From this it follows that for $k = |k_x| + |k_z| < 4$ the inequality (6.11) cannot be fulfilled for large enough $\mathscr{I}$, hence the theorem that follows.

In the presence of a stabilizing non-linearity, the amplitudes of the oscillations remain limited for any values of the distance from resonance and initial amplitudes if only the order of the resonance $k = |k_x| + |k_z|$ is smaller than 4. Only the character of the motion will depend on the fact that the initial amplitudes are large or small. In fact, it is not difficult to see that the wider the region where $\mathscr{I}$ varies, defined by (6.11), the smaller the stabilizing parameters ξ_x, ξ_z, ξ_{xz}, which, according to (6.9), are proportional to $I_i^{2-\frac{1}{2}k}$. For small initial amplitudes and when $k < 4$, these parameters are small, so that the beating ratio is large. If, however, one takes large initial amplitudes, the beating ratio is small. So, if the resonance order k is smaller than 4, oscillations of small amplitude (which are of general interest) will be unstable, in the sense that they will experience large beats. The amplitude of these beats, in its turn, will be limited by the stabilizing non-linearity. Thus the oscillation amplitudes remain limited for any distance from resonance.

The situation is somewhat different when $k > 4$. The smaller the initial amplitudes, the

larger the parameters $\xi_{x,z}$ and the smaller beats one will have, i.e. the closer the limits between which $\mathscr{I}$ can vary. In other words, the motion with small amplitudes around a resonance of high order ($k > 4$) is always stable. An unlimited increase of oscillation amplitude can only take place with sufficiently large initial amplitudes.

All these conclusions are the same as for the one-dimensional case studied in the preceding section. A more detailed investigation of the two-dimensional motion [18] shows that with a given stabilizing non-linearity the beat amplitude is usually of the same order as in the case of the one-dimensional oscillation.

7. APPLICATION OF THE THEORY OF NON-LINEAR EFFECTS TO PERIODIC MAGNET SYSTEMS

We shall now apply some results of the two preceding paragraphs to the way in which a cyclic accelerator works. We have shown there that all resonances can be divided into two classes according to their order. Resonances with $k < 4$ are characterized by the fact that small oscillations will be unstable in the sense that they will experience large beats (in the presence of a stabilizing non-linearity, of course, these beats will never go to infinity). The opposite picture is obtained with resonances of the fifth or higher orders: small oscillations remain stable and experience smaller beats, the less the initial amplitude. As the successful operation of accelerators depends mainly on the behaviour of just these small amplitudes, the lower order resonances (1st, 2nd, 3rd) are the dangerous ones, and one should choose the working point far away from them. Resonances of an order higher than the fourth apparently do not present a danger. In the neighbourhood of the fourth order resonance, the oscillations can be stable or unstable, depending on the practical parameters of the device.

To characterize the non-linearities of an ideal accelerator field, one can introduce the quantities

$$n_{k-1} = (-1)^k \frac{1}{K^k B_{0z}} \frac{\partial^k B_{0z}}{\partial x^k}, \tag{7.1}$$

which are in fact a generalization of the usual field index $n = n_0$. In strong focusing accelerators, where the resonance problems are particularly acute, the magnetic field depends strongly on the radius, and the quantities n_k generally satisfy the conditions

$$1 \ll |n| \ll |n_1| \ll |n_2| . \tag{7.2}$$

With these inequalities one succeeds in expressing the 'stabilizing' parameters (see 5.7 and 6.3), which describe the dependence of frequency on amplitude, in a rather compact form*

$$\begin{aligned} \zeta_{x,z} &= \frac{1}{4K_0^3 p}\langle K^4 n_2(\vartheta)|f_{x,z}(\vartheta)|^4\rangle , \\ \zeta_{xz} &= \frac{1}{K_0^3 p}\langle K^4 n_2(\vartheta)|f_x|^2|f_z|^2\rangle . \end{aligned} \tag{7.3}$$

We have already noticed that the coefficient ζ_{xz} takes, in some cases of practical interest, the value zero. This happens for instance in strong-focusing synchrotrons where the conditions for the x- and the z-motion are identical, i.e.

$$f_x^- \approx f_z^+ , \qquad f_x^+ \approx f_z^- ,$$

where the indices $-$ and $+$ refer to sectors with negative and positive field gradients, respectively. One can further admit that the parameter n_2, according to its definition (7.1) will have in $+$ and $-$ sectors the same absolute value but opposite signs. From this it follows that the coefficients ζ satisfy the approximate relations

$$\zeta_x \approx -\zeta_z , \qquad \zeta_{xz} \approx 0 . \tag{7.4}$$

Thus it appears that the x- and z-oscillations are independent, in this case, to a first approximation, even in the presence of a small non-linearity [6]. So, if one of the degrees of freedom possesses a well marked restoring force characteristic, i.e. the frequency varies when the amplitude increases, the other degree of freedom will have a less marked one. As we see from (7.3) the non-linear frequency variation with amplitude is given principally by parameter n_2 in this approximation, that is by the cubic field non-linearity.

In the course of acceleration, the form of the closed orbit and the magnetic field characteristics in it may vary periodically because of changes in momentum (synchrotron oscillations). This leads, of course,

* Because of the assumptions we have made, the formulae we shall give below are applicable mainly to strong-focusing ring accelerators. In some cases, depending on the particular properties of the machine, some of the terms we have neglected may be important. One must then go back to the complete expressions for the coefficients $u_{q,k,m}$.

to a change in betatron oscillation frequency. A large artificial non-linearity may aggravate this effect* as several resonance lines can be crossed a number of times by the working point during its motion. This circumstance must also be considered when one has to decide upon the permissible magnetic field non-linearity.

To evaluate the danger caused by the various resonances one should know the relationship between the various field perturbations and the expansion coefficients $u_{q,k,m}$.

In table 3 we have set down the expressions of the terms which give rise to resonance with perturbations. One should remember however that only the most important terms have been kept, as explained in the note on a previous page.

TABLE 3

Resonances	Perturbation exciting the resonance	Expression in terms of the azimuthal harmonic of the field perturbation
Linear		
1. Main $\nu_{x,z} = q$	$\lvert\langle u_{-q,1,1}\rangle\rvert$	$= \frac{1}{4\sqrt{2\pi}} \frac{e}{K_0 c}\left(\frac{2}{K_0 p}\right)^{\frac{1}{2}} \lvert(f^*_{x,z} b_{z,x})_q\rvert$
2. Parametric $\nu_{x,z} = \frac{1}{2}q$	$\lvert\langle u_{-q,2,2}\rangle\rvert$	$= \frac{1}{4\sqrt{2\pi}} \lvert(f^{*2}_{x,z} \Delta g_{x,z})_q\rvert$
3. Sum $\nu_x + \nu_z = q$	$\lvert\langle u_{-q,1,1,1,1}\rangle\rvert$	$= \frac{1}{4\sqrt{2\pi}} \frac{e}{K_0 c} \frac{2}{K_0 p} \left\lvert\left(f^*_x f^*_z \frac{\partial b_z}{\partial z}\right)_q\right\rvert$
Cubic		
4. $\nu_{x,z} = \frac{1}{3}q$	$\lvert\langle u_{-q,3,3}\rangle\rvert$	$= \frac{1}{24\sqrt{2\pi}} \frac{e}{K_0 c}\left(\frac{2}{K_0 p}\right)^{\frac{1}{2}} \left\lvert\left(f^{*3}_{x,z} \frac{\partial^2 b_{z,x}}{\partial(x,z)^2}\right)_q\right\rvert$
5. $2\nu_x + \nu_z = q$	$\lvert\langle u_{-q,2,1,2,1}\rangle\rvert$	$= \frac{1}{8\sqrt{2\pi}} \frac{e}{K_0 c}\left(\frac{2}{K_0 p}\right)^{\frac{1}{2}} \left\lvert\left(f^{*2}_x f^*_z \frac{\partial^2 b_x}{\partial x^2}\right)_q\right\rvert$
6. $\nu_x + 2\nu_z = q$	$\lvert\langle u_{-q,1,2,1,2}\rangle\rvert$	$= \frac{1}{8\sqrt{2\pi}} \frac{e}{K_0 c}\left(\frac{2}{K_0 p}\right)^{\frac{1}{2}} \left\lvert\left(f^*_x f^{*2}_z \frac{\partial^2 b_z}{\partial z^2}\right)_q\right\rvert$

* A deliberately introduced quadratic non-linearity, may, on the contrary, be useful in this respect, as it makes it possible to compensate the 'linear' frequency drift, (see ch. 4, 1.25).

With this table let us calculate, for example, the value of the stabilizing non-linearity which is necessary to suppress the cubic one-dimensional resonance caused by an azimuthal field asymmetry. If we require that condition $|\xi| > 20$ (see fig. 33) be fulfilled, and make use of expressions (5.15), (7.3) and table 3, we obtain

$$\left|\left(f_x^{*3}\frac{\partial^2 b_z}{\partial x^2}\right)_q\right| < 4\sqrt{2\pi}\,10^{-2}\,\frac{A|\langle B_z\rangle|}{|f|_{\max}K_0}\,|\langle K^4 n_2 |f_x|^4\rangle|\,,$$

where A is the amplitude of betatron oscillations. For a rough calculation let us suppose that the ideal equilibrium orbit is a circle, that is $K = K_0$. In strong-focusing synchrotrons with symmetrical sectors one can usually admit that the quantity n_1 is constant along the orbit, whereas n_2 has the same absolute value and opposite signs in neighbouring sectors. If the field perturbation b_z depends on the radius in the same way as the ideal field, one may write

$$\left|\left(f_x^{*3}\frac{\partial^2 b_z}{\partial x^2}\right)_q\right| \approx K_0^2 \nu_x^{-\frac{3}{2}} |n_1 b_{z(q)}|\,,$$

where $b_{z(q)}$ is the qth Fourier harmonic of the azimuthal asymmetry, and where the approximate value $f_x \approx \nu_x^{-\frac{1}{2}}$ has been used for the Floquet function (see ch. 2, 7.16). Simple computations show that for $K_0 A \approx 5\cdot 10^{-4}$ and $n_2/n_1 \approx 10 \div 100$ (which is generally the case in large strong-focusing synchrotrons), the tolerable field asymmetry $b_{z(q)}/B_{0z}$ should not exceed a value around one per cent. One sees from this that in an accelerator with a weak non-linearity only narrow resonance bands will be stabilized. In any case tolerances on the resonance perturbation harmonics remain extremely severe.

Apart from this, one should keep in mind that the parameters of betatron oscillations do not remain absolutely constant in practice, but always vary more or less rapidly. Variations of the distance δ from resonance, i.e. of betatron frequencies, are especially important as they may cause the working point to cross a resonance band. When this happens, the stabilizing action of the non-linearity can be practically destroyed due to certain phenomena which will be considered in the next section.

8. RESONANCES IN SYSTEMS WITH SLOWLY-VARYING PARAMETERS

We shall now examine in a semi-qualitative manner the phenomena occurring in systems whose parameters vary slowly compared with betatron or revolution frequencies [13, 17, 23]. With this assumption one may consider the magnetic system as periodic and apply some results from the preceding paragraph, in particular, make use of expression (5.10) for Hamilton's function. The most important change with respect to the static case is that the parameter δ is now dependent on ϑ so that the Hamiltonian, even in the variables (I, w) is no more an integral of the motion. We must stress that the equations of motion nevertheless keep their canonical form and that the (I, w) space is the phase space as before, although the phase trajectories do not in general coincide with the curves $\langle \mathscr{H}_1 \rangle = \text{const.}$, which have been drawn on figures 24, 26, 30, and 31.

Let us assume that a region G of the phase plane having a surface S is occupied by the representative points corresponding to all the particles in an accelerator. On the basis of Liouville's theorem on phase space conservation, one may then say that the surface S is an integral of the motion for any change of parameter δ. Also representative points, lying on the frontier of region G, will remain on this boundary however it may be deformed.

Let us consider now some closed curve $\langle \mathscr{H}_1 = \text{const.} \rangle$ in fig. 30 and let us assume that $\langle \mathscr{H}_1 \rangle$ varies little in a time corresponding to one period of revolution of the representative point on this curve in the static case. It is easily seen that this does not differ from a motion where the representative point would slowly go from one closed curve $\langle \mathscr{H}_1 \rangle =$ const. to another one. We shall call such conditions adiabatic variation of the parameters. Applying Liouville's theorem, we come to a conclusion sometimes known as the adiabatic theorem: when the parameters vary adiabatically, the representative points travel along closed paths which deform slowly, but in such a way that the surface they enclose remains constant. The mathematical formulation of this is the adiabatic invariance of the expression $\oint p\,\mathrm{d}q$, where p and q are the canonically conjugate variables, or in our case

$$\oint I\,\mathrm{d}w = \text{const.} \tag{8.1}$$

Another well defined case is the sudden change* of a parameter (for instance, the distance from resonance) in a time much shorter than the period of revolution around one phase trajectory. The phase diagram, in the course of such a jump, takes the form corresponding to the new parameter values, whereas the representative point continues to travel along the phase trajectory on which it was lying after the jump.

Let us apply these considerations to the crossing of resonances of various orders. If one looks at the phase diagram on fig. 30 one sees easily that a sudden frequency change, that is a fast crossing of the resonance, has little influence on particles with small oscillation amplitudes, since before and after the crossing, the representative points move along very close phase trajectories. The situation is quite different when the crossing occurs slowly. In fact, let some of the particles oscillate around the basic closed orbit, and others move along closed phase trajectories around the particular point a (see fig. 30b), which represents supplementary stable orbit appearing near resonance. Let us suppose now that the working point moves still further from resonance (fig. 30a). The supplementary orbits now move away from the ideal one, so that the particles which are captured in the side buckets will go away with them. We have mentioned already that the supplementary orbits correspond to oscillations which occur at synchronism with a perturbation, at a time when the oscillation frequency around the main orbit may already have moved away from the resonance value. Thus particles which fall into the side buckets are 'locked on' to the resonance and get lost when the working point moves away from the resonance region.

It is interesting to note that this process is basically asymmetric: when travelling through resonance in the opposite direction, the side buckets move towards the centre of the diagram. This corresponds exactly to the asymmetry of the resonance curve of the non-linear oscillator which we saw in § 5.

Thus it appears from qualitative considerations that by adiabatic changes of the resonance frequency the resonance band can be crossed in one direction, without the beating ratio becoming substantially larger than it was in the static case. We shall only remark that when the side buckets get near to each other, the particles previously lying in the

* A sudden variation means here a change which is fast compared with the motion of the representative point, but slow with respect to betatron oscillations.

centre bucket may partly fall into them and partly be swept into the external region, while the whole phase space occupied by them must remain constant. Thus an adiabatic crossing of a resonance even in the favourable direction increases the amplitudes of some of the particles at the cost of a reduction in the amplitude of the remaining ones, so that the total phase space occupied by the beam remains constant. When approaching the resonance from the other side, side buckets will show up near the centre (see fig. 30c) and take over a part of the particles. The remaining particles will be captured in the central bucket, which shows up soon afterwards, (fig. 30b) and do not increase substantially their oscillation amplitudes. It is difficult to indicate the exact dividing line between 'lock on' and passage through resonance. Accurate data can only be obtained by numerical integration of the equations of motion

$$I' = -\frac{\partial}{\partial w}\langle \mathscr{H}_1 \rangle = 2k|\langle u_{q,k,k}\rangle| I^{\frac{1}{2}k} \sin kw\,,$$

$$w' = \frac{\partial}{\partial I}\langle \mathscr{H}_1 \rangle = \delta(\vartheta) + 2\zeta I + k|\langle u_{q,k,k}\rangle| I^{\frac{1}{2}k-1} \cos kw\,. \tag{8.2}$$

taking into account the particular law of variation of $\delta(\vartheta)$. There is however a limiting value for the rate of change of δ, above which the particles cannot be captured into the side buckets and locked on in resonance. This is clear from physical considerations, as a given perturbation can only change the oscillation energy by a finite amount in finite time during which time the system has to be in resonance. In fact, it follows from (8.2) that the maximum rate of rise of the amplitude

$$|I'|_{\max} = \left|\frac{\partial\langle \mathscr{H}_1\rangle}{\partial w}\right|_{\max} = 2kI^{\frac{1}{2}k}|\langle u_{q,k,k}\rangle| \tag{8.3}$$

is obtained for $\sin kw = 1$, so that we have then

$$w' = 0 = \delta + 2\zeta I\,. \tag{8.4}$$

Thus a locking-on of the particles in resonance is impossible if δ increases more rapidly than the quantity I given by (8.3), that is if

$$\delta' > 4\zeta k I^{\frac{1}{2}k}|\langle u_{q,k,k}\rangle|\,. \tag{8.5}$$

Let us now examine the case at the other extreme: fast crossing of the resonance. Integrating the first of eqs. (8.2) we get

$$\mathscr{I}(\vartheta) = \begin{cases} \exp C \int_0^{\vartheta} \sin kw \, \mathrm{d}\vartheta , & k = 2 , \\ \left[1 - (\tfrac{1}{2}k - 1) C \int_0^{\vartheta} \sin kw \, \mathrm{d}\vartheta \right]^{\frac{2}{2-k}} , & k > 2 , \end{cases} \tag{8.6}$$

where the quantity $\mathscr{I}^{\frac{1}{2}}$ gives immediately the factor by which the amplitude has grown after the crossing, and where

$$C = 2k |\langle u_{q,k,k} \rangle| \, I_i^{\frac{1}{2}k-1} . \tag{8.7}$$

With the help of eqs. (8.2) one sees easily that the expression within the integral in (8.6) is a rapidly oscillating function of ϑ when $|\delta|$ is large. So the main contributions to the integrals come from the parts of the integration path which lay near the point of stationary phase ϑ_{st}, where $w' = 0$. In agreement with (8.2) we have at this point

$$w'' = \delta' + [2\zeta + k(\tfrac{1}{2}k - 1) |\langle u_{q,k,k} \rangle| \, I^{\frac{1}{2}k-2} \cos kw] \, 2k |\langle u_{q,k,k} \rangle| \, I^{\frac{1}{2}k} \sin kw \tag{8.8}$$

Let us assume that δ varies linearly with ϑ in the resonance region, and that the derivative δ' is rather large, in any case much larger than the other terms on the right-hand side of (8.8) (fast crossing of the resonance). Then $w'' \approx$ const., so that one may write in a wide range of the variable δ

$$w = w_{st} + w''|_{st} \tfrac{1}{2}(\vartheta - \vartheta_{st})^2 \approx w_{st} + \delta' \tfrac{1}{2}(\vartheta - \vartheta_{st})^2 . \tag{8.9}$$

With this expression one can now calculate the asymptotic value of the integral

$$\int_0^{\vartheta} \sin kw \, \mathrm{d}\vartheta \approx \int_0^{\vartheta} \sin k(w_{st} + \delta' \tfrac{1}{2}(\vartheta - \vartheta_{st})^2) \, \mathrm{d}\vartheta \approx \sqrt{\frac{\pi}{k|\delta'|}} \, [\sin kw_{st} \pm \cos w_{st} k] ;$$

$$|\delta'| \, \tfrac{1}{2}\vartheta_{st}^2 \gg 1 ; \quad |\delta'| \, \tfrac{1}{2}(\vartheta - \vartheta_{st})^2 \gg 1 , \tag{8.10}$$

where the upper or lower sign is to be taken depending on which of the inequalities $\delta' \gtrless 0$ applies.

Introducing expression (8.10) into (8.6) and considering that

$$|\sin\alpha \pm \cos\alpha| \leqslant \sqrt{2}$$

we obtain an upper limit for the increase of the amplitude after a fast passage through resonance

$$\mathscr{I}_{\max} = \begin{cases} \exp C\sqrt{\dfrac{\pi}{|\delta'|}}, & k = 2, \\ \left[1 - (\tfrac{1}{2}k - 1)C\sqrt{\dfrac{2\pi}{k|\delta'|}}\right]^{\frac{2}{2-k}}, & k > 2. \end{cases} \tag{8.11}$$

The coefficient of increase $\mathscr{I}_{\max}$ decreases when the crossing velocity $|\delta'|$ grows and tends towards 1 in the extreme case of a sudden frequency change ($\delta' \to \infty$). It should be remembered that expression (8.11) is generally applicable only for sufficiently rapid passages through resonance, when the first term on the right-hand side of (8.8) plays the major part.

Even fast passages through a resonance may lead to a considerable increase of amplitude if they repeat many times, which may happen when ν varies because of synchrotron oscillations. One can evaluate increases of this type on the basis of statistics, assuming that the stationary phase w_{st} takes random values at each of the crossings.

Examples of experimentally observed passages of particles through resonance are given in ch. 7, § 1. 3.

CHAPTER 4

SYNCHROTRON OSCILLATIONS

In the two preceding chapters we ignored completely the action of the electric field, assuming that it could only bring about slow changes of the parameters of motion, especially the energy. Regarding the results obtained up to now, the manner in which these changes take place does not appear to be essential. However, it is necessary to study them in detail, as upon them depends the function of the accelerator as a device for increasing the energy of particles.

In this chapter we will present the theory of the acceleration regime, that is the process by which the particle energy is varied under the action of the electric field. We shall pay special attention to the theory of self-focusing, as this principle is the basis of operation of the majority of present-day accelerators.

1. DEPENDENCE OF ORBIT PARAMETERS ON PARTICLE ENERGY

As we are going to study the change in energy of the particle, we have first to see how such a change influences the closed orbits. Let us consider again stationary conditions, that is a magnetic field constant in time, in which particles of different energies are moving. Let us assume that we have an orbit $r_0(\vartheta)$ corresponding to a particle momentum p. The question is, how much does the orbit of a particle with a momentum $p+\Delta p$ differ from it?

We shall consider in more detail the simple case of plane orbits. According to the general expressions (2.11) of ch. 2, the linear equation for the z-oscillation keeps its previous form for small changes Δp in momentum (ch. 2, formula 2.13b), whereas the equation for horizontal motion becomes inhomogeneous.

$$x''+g_x x = \frac{K}{K_0^2}\frac{\Delta p}{p}. \tag{1.1}$$

The appearance of an external force on the right is easily understood as the Lorentz and centripetal forces cannot be equal for particles with a momentum $p+\Delta p$ on an orbit other than their own. The resulting displacement of the equilibrium position leads of course to another closed orbit, corresponding to the momentum $p+\Delta p$.

The new closed orbit is evidently described by a particular periodic solution of (1.1) and according to general mathematical theorems [1] is unique when there is no resonance.

The solution is easily found by the method of variation of constants, taking the periodicity into account. We shall, however, use the solution (2.11) of ch. 3, obtained for the closed-orbit deviation caused by magnetic-field perturbations, in which we shall replace the perturbation $eb_z/K_0^2 pc$ with a period 2π by the quantity $K\Delta p/K_0^2 p$ with a period ϑ_0. We thus get

$$x_0(\vartheta) = \psi(\vartheta)\frac{\Delta p}{K_0 p}, \tag{1.2}$$

where

$$\psi(\vartheta) = \frac{\mathrm{i}}{2K_0}(1-\mathrm{e}^{-\mathrm{i}\nu\vartheta_0})^{-1}\int_{\vartheta}^{\vartheta+\vartheta_0} f^*(\vartheta')f(\vartheta)K(\vartheta')\mathrm{e}^{\mathrm{i}\nu(\vartheta-\vartheta')}\mathrm{d}\vartheta' + \text{c.c.} \tag{1.3}$$

Together with the envelope of betatron oscillations, the periodic function $\psi(\vartheta)$ plays an important role in accelerator theory and will often be encountered in the future.

Expression (1.2) should be interpreted as the deviation of the closed orbit due to the fact that the momentum differs from the nominal value. To get a clearer picture of its properties, ψ is sometimes expressed as a Fourier series with the phase variable $\hat{\vartheta}$ (see [2] and formula (3.6) of ch. 3).

$$\hat{\vartheta} = \frac{1}{\nu}\int\frac{\mathrm{d}\vartheta}{|f|^2}. \tag{1.4}$$

To do this, we can use again the expression (3.8) of ch. 3 where we replace $eb_z/K_0^2 pc$ by $K\Delta p/K_0^2 p$. This yields

$$\psi(\vartheta) = \frac{|f|\nu^2}{K_0}\sum_{k=-\infty}^{+\infty} C_k \frac{\mathrm{e}^{\mathrm{i}k\hat{\vartheta}}}{\nu^2-k^2}, \tag{1.5}$$

where the quantities C_k are the coefficients of the Fourier series

$$K(\vartheta)|f(\vartheta)|^3 = \sum_{k=-\infty}^{+\infty} C_k e^{ik\vartheta}. \tag{1.6}$$

According to (1.6), in a magnetic system with N periods the quantities C_k are different from zero only for $k = 0, \pm N, \pm 2N$, etc. In the majority of accelerators $K(\vartheta)$ and $|f(\vartheta)|$ do not differ very much from their mean value, so that the terms with $k = 0$ play the main role in the sums (1.5) and (1.6). This signifies that in these devices $\psi(\vartheta)$ is a positive function which does not change sign. In other words, the orbit corresponding to the momentum $p + \Delta p$ is always larger than that corresponding to p and completely envelopes it without intersecting it. Of course, the greater the orbit length under these circumstances, the higher will be p. This rule is not general however and may not be valid in machines where the orbit is far from circular. As one can see from (1.5), artificially produced orbit deviations with $k \approx \nu$ will have the strongest influence in this respect. We shall show in § 7 that such an arrangement of function $\psi(\vartheta)$ is in some cases appropriate.

Function $\psi(\vartheta)$ has its simplest form in the azimuthally symmetrical field

$$\psi = \frac{1}{1-n} = \text{const.} \tag{1.7}$$

According to (1.7), orbits corresponding to different energies are concentric circles of radius $R(E)$. In this case the simple relation

$$\frac{\Delta R}{R} = \frac{1}{1-n}\frac{\Delta p}{p} = \frac{1}{1-n}\frac{\Delta E}{\beta^2 E}, \tag{1.8}$$

holds, being valid for small $\Delta p/p$. It plays an important role in accelerator theory.

The function $\psi(\vartheta)$ has a much more complicated form for strong-focusing magnets, even in the simplest cases. (See ch. 7, § 2.) As an example function $\psi(\vartheta)$ is drawn in fig. 35 for an accelerator with equal sectors and without straight sections. As also the envelope of betatron oscillations, $\psi(\vartheta)$ has maxima in the sectors where there is radial focusing. This is a general rule as $\psi(\vartheta)$ satisfies the equation

$$\psi'' + g_x \psi = \frac{K}{K_0}. \tag{1.9}$$

As the right-hand side of (1.9) is positive and as $\psi'' < 0$ at the maximum, the maximum occurs for $g_x > 0$ (we assume that in this point $\psi > 0$).

In the slow processes occurring in accelerators, another quantity closely related to the function $\psi(\vartheta)$ also plays an important part. This parameter which one may call coefficient of orbit expansion* α reflects some

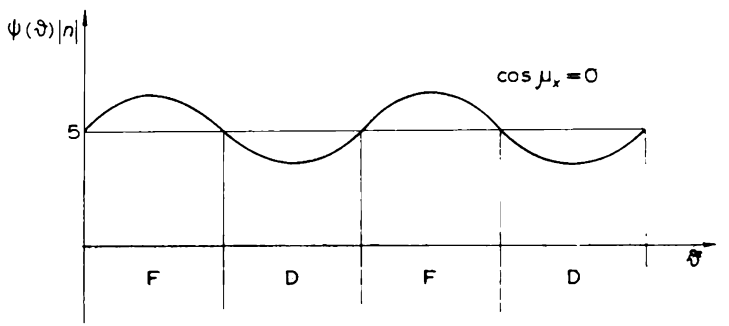

Fig. 35. The function $\psi(\vartheta)$ for a strong-focusing accelerator without straight sections.

average properties of the magnetic field in contrast with its local characteristics, given in ch. 2 and 3. By definition

$$\alpha = \left(\frac{\partial \ln \Pi}{\partial \ln p} \right)_B, \tag{1.10}$$

where Π is the orbit length of a particle with momentum p. The index B indicates that the derivative must be taken for a magnetic field constant in time. According to its definition the parameter α describes the relative change in orbit length when the momentum of the particle varies.

The quantity α can easily be related to the function $\psi(\vartheta)$. In fact, the quantity α is equal to

$$\Delta\Pi = \int_0^{2\pi} \frac{K x_0(\vartheta)}{K_0} \, \mathrm{d}\vartheta = \frac{\Delta p}{K_0^2 p} \int_0^{2\pi} K\psi \, \mathrm{d}\vartheta = \Pi \frac{\Delta p}{K_0 p} \langle K\psi \rangle, \tag{1.11}$$

* In foreign literature the term 'momentum compaction factor' is used.

so that

$$\alpha = \left\langle \frac{K}{K_0} \psi \right\rangle, \tag{1.12}$$

where the average must be taken along the orbit.

Making use of the Fourier expansion for function $\psi(\vartheta)$ one can bring α into the form

$$\alpha = \frac{\nu_x^3}{K_0^2} \sum_{k=-\infty}^{+\infty} \frac{|C_k|^2}{\nu_x^2 - k^2}. \tag{1.13}$$

Having written this expression, one can say the same as for (1.5): the main part is played by the term with $k = 0$, so that for machines where K and $|f|$ are nearly constant, one can consider that

$$\alpha \approx \nu_x^{-2}. \tag{1.14}$$

For the azimuthally symmetrical field, this approximation is fulfilled exactly.

Relation (1.14) gives a simple law for closed orbits: the stronger the radial focusing, that is the larger ν_x, the closer to each other in the machine are packed the orbits corresponding to various energies at one precise instant. These orbits may in general be different in form.

Rather similar considerations apply to the case of three-dimensional orbits. The only difference is that function $\psi(\vartheta)$ appears now as a vector having x- as well as z-components. The orbit corresponding to the momentum $p + \Delta p$ can be written in the form

$$\boldsymbol{r}_0(\vartheta, p + \Delta p) = \boldsymbol{r}_0(\vartheta, p) + \boldsymbol{\psi}(\vartheta) \frac{\Delta p}{K_0 p}. \tag{1.15}$$

The relative lengthening of the perimeter is as before

$$\alpha = \frac{p}{\Pi} \left(\frac{\partial \Pi}{\partial p} \right)_B = \left\langle \frac{K}{K_0} \boldsymbol{\psi} \boldsymbol{n} \right\rangle, \tag{1.16}$$

where $\boldsymbol{n}$ is the unit vector along the external normal.

In the same way the dependence of the orbit on the value of the magnetic field on the initial orbit can be found. According to (1.9) of ch. 2,

$$\frac{\mathrm{d}K}{K} = \frac{\mathrm{d}B_z}{B_z} - \frac{\mathrm{d}p}{p}. \tag{1.17}$$

We are interested in such changes of field where the field configuration remains the same whereas only the absolute value of the field varies. For orbit calculations the relative field increase $-(\mathrm{d}B_z\, B_z)$ is, as one sees from (1.17), completely equivalent to the relative change in momentum $\mathrm{d}p/p$. So we obtain

$$x_0(\vartheta) = -\psi(\vartheta)\,\frac{\mathrm{d}B_z}{B_z K_0} \tag{1.18}$$

and

$$\frac{\langle B_z\rangle}{\Pi}\left(\frac{\partial \Pi}{\partial\langle B_z\rangle}\right)_p = -\frac{p}{\Pi}\left(\frac{\partial \Pi}{\partial p}\right)_B = -\alpha\,. \tag{1.19}$$

The last relation can also be obtained immediately from eq. (1.11) of ch. 2.

The form of the closed orbit and the characteristics of the magnetic field on it determine the betatron oscillation frequencies, and finally the stability of the motion. As particles always have some momentum spread in an accelerator, their betatron frequencies will be somewhat different. It is easy to find out this dependence of the frequencies on momentum, making use of the results of the preceding chapters. In fact, a momentum change Δp will cause on a given azimuth a change in the quantities $g_{x,z}$, which enter into the equations of motion (2.13) of ch. 2:

$$\begin{aligned} \Delta g_x &= \Delta\left[\frac{K^2}{K_0^2}(1-n)\right], \\ \Delta g_z &= \Delta\left[\frac{K^2}{K_0^2}\,n\right]. \end{aligned} \tag{1.20}$$

If one restricts oneself to linear terms in Δp, one can write

$$\begin{aligned} K(p+\Delta p) &\approx K(p) + \left(\frac{\partial K}{\partial p} + \frac{\partial K}{\partial x}\,\frac{\partial x_0}{\partial p} - \frac{\partial K}{\partial \vartheta}\,\frac{\partial \vartheta}{\partial p}\right)\Delta p\,, \\ K_0(p+\Delta p) &\approx K_0(p) + \frac{\partial K_0}{\partial p}\,\Delta p\,, \\ n(p+\Delta p) &\approx n(p) + \left(\frac{\partial n}{\partial p} + \frac{\partial n}{\partial x}\,\frac{\partial x_0}{\partial p} - \frac{\partial n}{\partial \vartheta}\,\frac{\partial \vartheta}{\partial p}\right)\Delta p\,. \end{aligned} \tag{1.21}$$

The last terms in (1.21) containing $\partial\vartheta/\partial p$ appear because the quantity ϑ varies when one goes from a point on the initial orbit along the normal to the point on the new orbit whereas the expansion is made for $\vartheta =$ const. The derivatives appearing in (1.21) may be found by using expressions (1.9), (2.12) of ch. 2, (1.2) and (1.10) and also the relationship between the arc lengths $\sigma(p+\Delta p)$ and $\sigma(p)$:

$$\sigma(p+\Delta p) = \sigma(p) + \frac{\Delta p}{p} \int_0^{\sigma(p)} \frac{K}{K_0} \psi \, \mathrm{d}\sigma \,. \tag{1.22}$$

We finally obtain

$$\Delta g_x(\vartheta) = \frac{\Delta p}{p} \frac{K^2}{K_0^2} \left\{ -2+n+2\alpha(1-n) - (2n-n_1) \frac{K\psi}{K_0} - \right.$$
$$\left. - \left[2(1-n) \frac{K'}{K} - n' \right] \int_0^{\vartheta} \left(\frac{K\psi}{K_0} - \alpha \right) \mathrm{d}\vartheta \right\}, \tag{1.23}$$
$$\Delta g_z(\vartheta) = \frac{\Delta p}{p} \frac{K^2}{K_0^2} \left\{ -n+2n\alpha - n_1 \frac{K\psi}{K_0} - \left[\frac{2nK'}{K_0} + n' \right] \int_0^{\vartheta} \left(\frac{K\psi}{K_0} - \alpha \right) \mathrm{d}\vartheta \right\},$$

where n_1 is the non-linear field characteristic (see (7.1) of ch. 3).

According to expression (2.7) of ch. 3, the frequency increases we are looking for are expressed by formula

$$\Delta \nu_{x,z} = \tfrac{1}{2} \langle |f_{x,z}|^2 \Delta g_{x,z} \rangle \,, \tag{1.24}$$

where $f_{x,z}$ is the Floquet function describing the betatron oscillations around the initial orbit*.

In the case of strong-focusing accelerators, the major contribution to the quantities $\Delta g_{x,z}$ usually comes from the terms $\propto n(\vartheta)$ and $\propto n_1$ (see 1.23), so that their order of magnitude is

$$\Delta \nu_{x,z} \approx \pm \frac{1}{2} \frac{\Delta p}{p} \left\langle |f_{x,z}|^2 \left(n + n_1 \frac{K\psi}{K_0} \right) \right\rangle \,. \tag{1.25}$$

The quantity (1.24) must be much smaller than the dimensions of the

* In accelerators with 'scaling' orbits (ch. 7, § 3) the betatron oscillation frequencies $\nu_{x,z}$ do not depend on the momentum p, which corresponds to $\Delta g_{x,z} \equiv 0$.

resonance cell, which imposes additional limitations on $\Delta p/p$. This requirement is especially important in strong-focusing synchrophasotrons where it seriously limits the permissible energy spread in the injected beam.

2. BASIC EQUATION OF THE ACCELERATION REGIME

In the space where the motion of the particle takes place there is, besides the magnetic field $\boldsymbol{B}$, also an electric field $\boldsymbol{\mathscr{E}}$, so that the equation of motion (1.7) of ch. 2 has to be supplemented:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) = \frac{e}{c}\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\,\boldsymbol{B}(\boldsymbol{r},t)\right] + e\boldsymbol{\mathscr{E}}(\boldsymbol{r},t)\,, \tag{2.1}$$

where we assume that the magnetic field $\boldsymbol{B}$ and the electric field $\boldsymbol{\mathscr{E}}$ depend explicitly upon time. In this chapter we will be interested not so much in the position of the particles as in their change in energy, which, as we know, is described by the equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = e\boldsymbol{\mathscr{E}}(\boldsymbol{r},t)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\,, \tag{2.2}$$

where $E = mc^2$ is the total particle energy.

The right-hand side of eq. (2.2) is, in general, a rapidly changing function of time. For this reason, besides its slow systematic variation, the energy also experiences small oscillations which average out to zero. These small oscillations do not present particular physical interest, and are not usually considered. As far as the systematic variation $E(t)$ is concerned one can obtain its law by the averaging method, taking the mean value of the right-hand side of eq. (2.2) with respect to time, and assuming the energy E to remain constant. Of course, in doing this, one has to take into account also the non-explicit dependence of the field on time through the co-ordinate of the particle.

The motion with constant energy has already been studied in detail in the preceding chapters. As we showed there, the radius vector may be written in the form

$$\boldsymbol{r}(t) = \boldsymbol{r}_0(t) + \boldsymbol{\rho}(t)\,, \qquad |\boldsymbol{\rho}| \ll |\boldsymbol{r}_0|\,, \tag{2.3}$$

where r_0 corresponds to the closed orbit and ρ to the betatron oscillations. Let us observe now that $r_0(t)$ depends on t in a periodic manner with the frequency*

$$\omega(E,t) = \frac{2\pi v(E)}{\Pi(E,t)}, \tag{2.4}$$

whereas the betatron oscillation frequency is equal to $v\omega(E,t)$. It is important that these frequencies are not harmonically related, as the working point of the accelerator has been assumed to lie sufficiently far from resonance.

In accelerator practice the most interesting cases are those where the electric field on the orbit is either quasi-stationary or has the form of harmonic standing waves. So we shall suppose that

$$\mathscr{E}(r,t) = \mathscr{E}(r)\,e^{-i\int\omega_0 dt} + \mathscr{E}_{ind}(r)\,, \tag{2.5}$$

where the frequency of the electric field (the so-called radiofrequency) ω_0 and the amplitude $\mathscr{E}(r)$, so as $\mathscr{E}_{ind}(r)$ may vary slowly with time, that is change little in the course of a revolution along the orbit $2\pi/\omega$. In resonance accelerators of the phasotron or synchrotron type it is the radiofrequency component $\mathscr{E}(r)e^{-i\int\omega_0 dt}$ which plays the fundamental role, whereas in betatrons it is the field $\mathscr{E}_{ind}$, which for this reason is sometimes called the 'betatron' field.

With these conceptions let us take the time average equation (2.2). If we introduce expression (2.3) in place of $r(t)$ and if we linearize with respect to ρ, we find easily that all linear terms in ρ or $d\rho/dt$ disappear unless the resonance condition for betatron oscillations is fulfilled. In fact, we have to take the time average of terms like

$$\mathscr{E}(r_0)\,e^{-i\int\omega_0 dt}\rho(t)\,, \tag{2.6}$$

where the first factor has the frequency ω, the second $-\omega_0$, and the third $-\nu_{x,z}\omega$. As in the average $\langle\rho\rangle = 0$, the average of the whole expression is equal to zero. Thus the energy variation is independent of betatron oscillations** and given by the equation

$$dE/dt = ev(E)\langle\mathscr{E}_\tau(r_0(t,E),t)\rangle\,, \tag{2.7}$$

* The explicit dependence of the revolution frequency upon time is related to the slow variation of the magnetic field averaged along the orbit.

** This is not so near a resonance where the frequencies ω and $v\omega$ are harmonically related. This case is not particularly interesting and will not be considered.

where $\mathscr{E}_\tau$ is the $\mathscr{E}$ component of the vector which lies tangential to the orbit, that is $\mathscr{E}_\tau = \mathscr{E}_t$. In passing from (2.2) to (2.7) we have used the relation

$$\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}\vartheta}\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = v(E)\boldsymbol{\tau}\,. \tag{2.8}$$

The physical sense of the mean values equation (2.7) lies in the fact that the slow variation of energy is determined by the electric field averaged over the instantaneous orbit.

The change of the main magnetic field of the accelerator occurs slowly, adiabatically, with respect to betatron oscillations. It has to be taken into account only when considering slow processes. Also, if one uses a high-frequency field for acceleration, as in resonance accelerators, there is a magnetic high-frequency field coupled with it. Although this is usually small it is advisable to take it into account in special cases, as will be seen below (see § 3).

The mean value of the electric field along the orbit depends, of course, on the dimensions and the form of the orbit and consequently on the particle energy and the magnetic guiding field. This dependence, which determines the features of the acceleration regime, has a different character in devices of different types.

3. RESONANCE ACCELERATION REGIME. PHASE EQUATION

In the majority of present-day accelerators the electric field has the form (2.5) and its frequency is close to an integral multiple of the particle revolution frequency $\omega_0 \approx q\omega(E, t)$, where the integral number q is called the 'harmonic' of the acceleration. It is well known that it is just this harmonic relationship between revolution frequency and radiofrequency which permits the resonance regime of acceleration in cyclic accelerators (see ch. 1 § 5).

We give the name 'equilibrium' or 'synchronous' particle to a particle whose revolution frequency is at every instant an exact sub-multiple of the high frequency ω_0. The corresponding energy and orbit will be called 'equilibrium' energy and orbit and characterized by the index s.

Thus,
$$\omega_s(t) = \frac{\omega_0(t)}{q} = \omega(E_s(t), t)\,. \tag{3.1}$$

The second part of this relation defines the equilibrium energy $E_s(t)$. Usually this definition is unique. However, cases can occur where, for a given radiofrequency, several equilibrium energies are possible, corresponding for instance to different harmonic numbers q. Such a possibility will be considered in ch. 7. Until then we shall assume that at any moment there can be only one equilibrium orbit in the vacuum chamber of the accelerator, and that the harmonic number q does not depend on time.

According to the definition, the equilibrium particle is always in exact synchronism with the accelerating field and the energy it receives per unit time is

$$\frac{dE_s}{dt} = \frac{E_s}{\mathscr{K}}\left[\frac{1}{\omega_s}\left(\frac{\partial\omega}{\partial t}\right)_s - \frac{1}{\omega_0}\frac{d\omega_0}{dt}\right], \tag{3.2}$$

where the quantity

$$\mathscr{K} = -\frac{E_s}{\omega_s}\left(\frac{\partial\omega}{\partial E}\right)_s \tag{3.3}$$

is an important characteristic for the self-focusing properties of the accelerator. Let us note that parameter $\mathscr{K}$ is easily expressed in terms of α. In fact

$$\frac{\partial\omega}{\partial E} = \frac{2\pi}{\Pi}\frac{dv}{dE} - \frac{2\pi v}{\Pi^2}\left(\frac{\partial\Pi}{\partial E}\right)_B = \frac{2\pi}{\Pi}\frac{dv}{dE} - 2\pi\frac{dp}{dE}\frac{v\alpha}{\Pi p} = \frac{\omega}{E}\frac{1-\alpha\gamma^2}{\gamma^2-1},$$

so that

$$\mathscr{K} = \frac{\alpha\gamma_s^2-1}{\gamma_s^2-1}, \qquad \gamma_s = \frac{E_s}{m_0c^2}. \tag{3.4}$$

As $\omega(E,t) = 2\pi\nu(E)/\Pi(E,t)$ one may bring (3.2) into the form

$$\frac{1}{E_s}\frac{dE_s}{dt} = \frac{1}{\mathscr{K}}\left[\frac{\alpha}{\langle B_z\rangle_s}\frac{\partial\langle B_z\rangle_s}{\partial t} - \frac{1}{\omega_0}\frac{d\omega_0}{dt}\right], \tag{3.5}$$

where we made use of relation (1.19). Thus it is essential, if the equilibrium energy is to be varied, that either the radiofrequency or the

magnetic field of the accelerator should vary*. The first of these possibilities, as shown in ch. 1, is called phasotron regime, the second one synchrotron regime. It is possible to have the two mixed together, when ω_0 and $\langle B_z \rangle$ vary at the same time; usually this is termed the synchrophasotron regime. An exception to this rule is the isochronous cyclotron, where the revolution frequency does not depend at all on energy, $\partial\omega/\partial E = 0$. Here the concept of equilibrium particle is not applicable.

The relative rates of change of the magnetic field and the radiofrequency differ from one type of accelerator to another. In particular, in annular cyclic accelerators it is convenient to keep the equilibrium orbit always in the same place. It is easily found from (3.5) and (1.9) of ch. 2 that the rates of change of magnetic field and radiofrequency are connected by the relation

$$\frac{1}{\langle B_z \rangle_s} \frac{\partial \langle B_z \rangle_s}{\partial t} = \frac{\gamma_s^2}{\omega_0} \frac{\mathrm{d}\omega_0}{\mathrm{d}t} . \tag{3.6}$$

From this it is seen, incidentally, that the radiofrequency may be practically constant in electron accelerators ($\gamma_s \gg 1$); but must be varied considerably in heavy particle accelerators ($\gamma_s \gtrsim 1$) of the synchrophasotron type.

Let us now return to the basic energy relation (2.7) and look at the operation of averaging the high-frequency electric field $\mathscr{E}(r_0)\mathrm{e}^{-\mathrm{i}\int\omega_0 \mathrm{d}t}$. The position of a particle is characterized by two quantities: its energy E which determines the orbit, and the generalized azimuth ϑ. Hence $\mathscr{E}_\tau(r_0) = \mathscr{E}_\tau(E, \vartheta)$ is evidently a periodic function of ϑ which may be expanded in a Fourier series

$$\mathscr{E}_\tau(r_0) = \sum_{k=-\infty}^{+\infty} \varepsilon_k(E)\,\mathrm{e}^{\mathrm{i}k\vartheta} . \tag{3.7}$$

Observing that

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \omega(E, t) , \tag{3.8}$$

* We consider an explicit dependence of the magnetic field on the orbit upon time, so that $\langle B_z \rangle$ may vary also when the particle energy varies: $\mathrm{d}/\mathrm{d}t\,\langle B_z \rangle \neq \partial/\partial t\,\langle B_z \rangle$. One has to keep in mind that, if the energy remains constant, a change in the field produces a displacement of the orbit (see 1.19). This is why the coefficient α appears in eq. (3.5).

we then obtain

$$\langle \mathscr{E}_\tau(r_0) e^{-i\int \omega_0 dt} \rangle = \sum_{k=-\infty}^{+\infty} \varepsilon_k \langle \exp i\int (k\omega - \omega_0) dt \rangle . \tag{3.9}$$

According to the physical principle of resonance acceleration, the radio-frequency is approximately q times the revolution frequency: $\omega_0 = q\omega$ (for the equilibrium particle this equation is fulfilled exactly). Thus the quantities on the right-hand side which are to be averaged are, in general, fast oscillating functions of time, and give an average of zero. An exception is the resonance term with $k = q$ as, according to (3.1), its frequency is close to zero. So we get

$$\langle \mathscr{E}_\tau(r_0) e^{-i\int \omega_0 dt} \rangle \simeq \varepsilon_q e^{i\int (q\omega - \omega_0) dt} , \tag{3.10}$$

or, considering that it is the real part of this quantity that has a physical meaning

$$\mathrm{Re} \langle \mathscr{E}_\tau(r_0) e^{-i\int \omega_0 dt} \rangle \simeq \frac{V \cos \varphi}{\Pi(E)} . \tag{3.11}$$

Here we have introduced a new quantity φ defined by

$$\frac{d\varphi}{dt} = q(\omega_s - \omega) . \tag{3.12}$$

Also, we have used for convenience the expression $V = |\varepsilon_q| \Pi(E)$.

The physical meaning of the quantity φ is extremely important. The expression of the electric field in the form of the series (3.7) signifies that we expanded it in terms of waves of frequency ω_0 travelling along the azimuth with a wave-number k. Of all these, only the wave which is nearly in resonance with the particle (or which has a phase velocity close to the velocity of the particle) can exert a systematic action on it. The quantity φ represents the phase which the particle has with respect to this wave. The significance of the quantity eV becomes clear also: it is the maximum energy which the travelling wave in resonance can communicate to the particle in one revolution (with the phase $\varphi = 0$). The quantity V is usually called 'accelerating voltage' in analogy to the case where the accelerating device consists of an infinitely narrow gap, to which a harmonic voltage of amplitude V is applied. In this special case

the meaning of φ is particularly evident: it is the phase of the voltage at the moment when the particles pass through the gap.

If the magnetic guiding field of the accelerator depends explicitly on time there exists in addition to the high-frequency field also a quasi-stationary field $\mathscr{E}_{\tau\,\mathrm{ind}}$, for which

$$\langle \mathscr{E}_{\tau\,\mathrm{ind}} \rangle = -\frac{1}{c\Pi(E,t)} \frac{\partial \Phi}{\partial t}, \tag{3.13}$$

where Φ is the magnetic flux enclosed by the orbit. Introducing the expressions (3.13) and (3.11) into the equation of electrical balance (2.7) we obtain the relation

$$\frac{1}{\omega} \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{eV}{2\pi} \cos\varphi - \frac{e}{2\pi c} \frac{\partial \Phi}{\partial t}, \tag{3.14}$$

which shows that the whole mean energy change of the particle in the course of one revolution is divided into one part due to the action of the resonance wave and another part resulting from the explicit time dependence of the magnetic field.

Let us now define the equilibrium phase φ_s with the help of the relation

$$\cos\varphi_s = \frac{2\pi}{eV}\left[\frac{1}{\omega_s}\frac{\mathrm{d}E_s}{\mathrm{d}t} + \frac{e}{2\pi c}\frac{\partial \Phi_s}{\partial t}\right], \tag{3.15}$$

where the quantity $\mathrm{d}E_s/\mathrm{d}t$ is found from the law of variation with time of the radiofrequency and magnetic field on the equilibrium orbit (see 3.5). If $|\cos\varphi_s| \leqslant 1$ then φ_s has the physical meaning of the phase of the equilibrium particle. If however $|\cos\varphi_s| > 1$, it means that the accelerating field is not large enough to ensure the increase of the equilibrium energy given by the law of variation of $\omega_0(t)$ and $\langle B_z \rangle_s$. When $|\cos\varphi_s| > 1$, self-focusing is impossible.

It is natural that in accelerators with time-varying magnetic field (synchrotron, synchrophasotron) the equilibrium phase should be determined by the accelerating voltage and the rate of increase of the magnetic field. In fact, making use of (1.11) of ch. 2 and of (3.15), we obtain

$$\cos\varphi_s = \frac{2\pi}{eV}\left[\frac{e}{K_0^2 c}\frac{\partial |\langle B_z \rangle|_s}{\partial t} + \frac{e}{2\pi c}\frac{\partial \Phi_s}{\partial t}\right] \approx \frac{2\pi}{K_0^2 cV}\frac{\partial |\langle B_z \rangle|_s}{\partial t} \tag{3.16}$$

where the second part of the equation is valid in the case which is most important in practice, where the high frequency accelerating tension exceeds largely the betatron e.m.f.

In accelerators with a guiding field constant in time (phasotron, annular phasotron) we obtain a simple expression for $\cos \varphi_s$ from (3.15) and (3.5)

$$\cos \varphi_s = -\frac{2\pi E_s}{qeV\omega_s^2 \mathcal{K}} \frac{d\omega_0}{dt}. \tag{3.17}$$

Besides the motion of the equilibrium particle, it is important to know the behaviour of particles with energy different from E_s, which is described, evidently, by the following pair of equations

$$\frac{1}{\omega}\frac{dE}{dt} - \frac{1}{\omega_s}\frac{dE_s}{dt} = \frac{eV}{2\pi}(\cos\varphi - \cos\varphi_s) - \frac{e}{2\pi c}\frac{\partial \Delta\Phi}{\partial t}, \tag{3.18a}$$

$$\frac{d\varphi}{dt} = q(\omega_s - \omega). \tag{3.18b}$$

Equations (3.18) are valid for any energy difference from the equilibrium value (of course, with the assumption made above that there exists only one equilibrium orbit and that the motion is determined by only one travelling wave of harmonic number q). In the form they have, eqs. (3.18) are adequate only to clear some special questions of acceleration in a constant field (see ch. 7). In most practical cases one would like to know the behaviour of particles whose energy differs little from the equilibrium one, so that one may linearize eqs. (3.18) with respect to $\Delta E/E_s$. In doing this, we shall use the following relations, which are valid in the approximation linear in ΔE

$$\frac{1}{\omega}\frac{dE}{dt} - \frac{1}{\omega_s}\frac{dE_s}{dt} = \frac{1}{\omega_s}\frac{d\Delta E}{dt} - \frac{\Delta E}{\omega_s^2}\left(\frac{\partial\omega}{\partial E}\right)_s \frac{dE_s}{dt},$$

$$\frac{d\varphi}{dt} = q(\omega_s - \omega) = q\omega_s \mathcal{K}\frac{\Delta E}{E_s}, \tag{3.19}$$

where the partial derivative $(\partial\omega/\partial E)_s$ is taken on the equilibrium orbit and calculated with a magnetic field invariant with time. In addition

it is necessary to express in terms of ΔE the difference of rotational e.m.f. acting on the non-equilibrium and the equilibrium particles. For this we shall use the relation

$$\frac{e}{2\pi c}\frac{\partial \Delta \Phi}{\partial t} = \frac{e}{2\pi c}\frac{\Delta E}{\beta_s^2 E_s K_0^2}\oint \frac{\partial B_{zs}}{\partial t}\psi(\vartheta)\mathrm{d}\vartheta\,. \tag{3.20}$$

As the configuration of the magnetic field does not vary noticeably in the course of the acceleration, one may take the quantity

$$\frac{1}{B_{zs}}\frac{\partial B_{zs}}{\partial t} = \frac{1}{\langle B_z\rangle_s}\frac{\partial \langle B_z\rangle_s}{\hat{c}t}$$

before the integral sign, after which (3.20) will give, taking into account (1.19),

$$\frac{e}{2\pi c}\frac{\partial \Delta \Phi}{\partial t} = -\frac{\alpha}{\omega_s}\Delta E\frac{1}{\langle B_z\rangle_s}\frac{\hat{\partial}\langle B_z\rangle_s}{\partial t} = -\frac{\Delta E}{\omega_s^2}\left(\frac{\hat{c}\omega}{\hat{c}t}\right)_{E_s} \tag{3.21}$$

Introducing (3.19) and (3.21) into eq. (3.18), the following system is obtained

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta E}{\omega_s}\right) &= \frac{eV}{2\pi}(\cos\varphi - \cos\varphi_s)\,,\\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} &= q\omega_s\mathscr{K}\frac{\Delta E}{E_s}\,.\end{aligned} \tag{3.22}$$

Eliminating ΔE from (3.22) we obtain the 'phase equation', which describes slow changes of phase and energy [3–6],

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{E_s\mathrm{d}\varphi/\mathrm{d}t}{\omega_s^2\mathscr{K}}\right] = \frac{qeV}{2\pi}(\cos\varphi - \cos\varphi_s)\,. \tag{3.23}$$

It is easily seen that if the parameters $\cos\varphi_s$, V, ω_s and $\mathscr{K}$ vary slowly enough, as is usually the case in practice, eq. (3.23) describes an oscillatory motion taking place in a potential well of the form

$$\frac{qeV}{2\pi}\int(-\cos\varphi + \cos\varphi_s)\mathrm{d}\varphi\,.$$

We should stress, however, that the derivation given above may turn

out to be not sufficiently general. In particular we assumed implicitly that the accelerating voltage amplitude V does not depend explicitly on energy, that the revolution frequency depends on the energy but not on the phase, etc. However, one can prove that the phase equation has the form (3.23) with much more general assumptions than we made for its derivation [7].

For example, let us consider the phase motion with any arbitrary high-frequency field, assuming only that its frequency is close to a multiple of the particle revolution frequency. We will designate by $\mathscr{E}(\vartheta, \rho, t)$ and $\boldsymbol{h}(\vartheta, \rho, t)$ the electric and magnetic high-frequency fields respectively, where ρ is the vector perpendicular to the equilibrium orbit at point ϑ. Making use of the concept of particle phase φ, one may eliminate the explicit dependence of these fields on time, by writing $t = t(\vartheta, \varphi)$. There acts on a non-equilibrium particle at a point of the equilibrium orbit, in addition to the force $(K/K_0^2)(\Delta p/p_s)\boldsymbol{n}^*$, yet another force due to the high-frequency field

$$\boldsymbol{F}(\vartheta, \varphi) = \frac{e}{\beta_s^2 E_s K_0^2} [\mathscr{E}_\perp(\vartheta, \varphi) + \beta_s[\boldsymbol{\tau} \boldsymbol{h}(\vartheta, \varphi)]], \tag{3.24}$$

$$\boldsymbol{F}(\vartheta, \varphi_s) = 0,$$

where the sign $\perp$ means that one should take only the components of the particular vector which are perpendicular to the orbit. The full excursion of the closed orbit from its equilibrium position may consequently be expressed in the form

$$\rho_0(\vartheta, \varphi) = \psi \frac{\Delta p}{K_0 p_s} + \rho_{HF}(\vartheta, \varphi), \qquad \rho_{HF}(\vartheta, \varphi_s) \equiv 0, \tag{3.25}$$

where ρ_{HF} is the additional excursion due to the high-frequency field. It is important to notice that ψ and ρ_{HF} satisfy the same linear equation, but with different right-hand sides, so that

$$L\psi = \frac{K}{K_0} \boldsymbol{n}, \qquad L\rho_{HF} = \boldsymbol{F}(\vartheta, \varphi), \tag{3.26}$$

where L is a differential linear operator of the second order with periodic coefficients (see ch. 2, eq. 2.14). From the energy balance we obtain in the linear approximation in ΔE, instead of (3.4),

$$\frac{1}{\omega} \frac{dE}{dt} = \frac{e}{2\pi} U(E, \varphi) - \frac{e}{2\pi c} \frac{\partial \Phi}{\partial t}, \tag{3.27}$$

where U is the high-frequency e.m.f. in one revolution

$$U(E, \varphi) = \oint \mathscr{E}(\vartheta, \varphi) d\sigma, \tag{3.28}$$

* This quantity does not have, of course, the dimension of a force. We call it in this way only because it appears on the right-hand side of an equation of the type (1.1).

where the integration has to be carried out along the orbit corresponding to the energy E. Let us note that we deliberately did not represent $U(E, \varphi)$ in the form $V(E)\cos\varphi$, since in the most general case the accelerating field may well depend upon time in a non-harmonic manner as we have only assumed its periodicity. As regards the explicit dependence of the accelerating voltage on particle energy, it can be calculated from (3.27).

Let us write the phase equation in the form

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = q(\omega_s - \omega) = q\omega_s \left\langle \frac{\Delta E}{\gamma_s^2 \beta_s^2 E_s} - K\rho \boldsymbol{n} \right\rangle = q\omega_s \mathscr{K} \frac{\Delta E}{E} + q\omega_s \langle K\rho_{\mathrm{HF}} \boldsymbol{n} \rangle . \quad (3.29)$$

where the second term describes the dependence of the frequency on the phase of the particle. Physically this dependence is due to the presence of radial electric and vertical magnetic high-frequency fields, which change the revolution frequency of the particle.

Eliminating ρ_0 and ΔE from (3.27), (3.29) and (3.25) and neglecting quadratic terms of the high-frequency field we obtain the phase equation in its general form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{E_s \mathrm{d}\varphi/\mathrm{d}t}{\omega_s^2 \mathscr{K}}\right] - \Gamma(\varphi, t)\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{qe}{2\pi}\left[U(E_s, \varphi) - U(E_s, \varphi_s)\right], \quad (3.30)$$

where

$$\Gamma(\varphi, t) = \frac{E_s}{\omega_s \mathscr{K}}\left[\frac{e}{2\pi c \Delta E}\frac{\partial \Delta \Phi}{\partial t} - \frac{\mathrm{d}\omega_s}{\mathrm{d}t}\frac{1}{\omega_s^2} + \frac{e}{2\pi}\left(\frac{\partial U}{\partial E}\right)_{E_s} + q\frac{\mathrm{d}}{\mathrm{d}\varphi}\langle K\rho_{\mathrm{HF}} \boldsymbol{n}\rangle\right] . \quad (3.31)$$

At first glance it may be thought that the phase equation (3.30) describes a motion which is damped because of the presence of the dissipative term $\Gamma(\varphi, t)\mathrm{d}\varphi/\mathrm{d}t$. It can be proved, however, that in all systems without exception $\Gamma(\varphi, t) \equiv 0$.

To prove this, we note that in any system the following equality holds

$$\langle \rho_{\mathrm{HF}} L\psi \rangle = \langle \psi L\rho_{\mathrm{HF}} \rangle . \quad (3.32)$$

As an example, let $V_i(\vartheta)$ be the eigenfunctions of an operator L, which are orthogonal in the interval 0 to 2π and have eigenvalues λ_i, respectively. Expanding ψ and ρ_{HF} in terms of the complete system V_i and taking the orthogonality property into account, we obtain

$$\langle \rho_{\mathrm{HF}} L\psi \rangle - \langle \psi L\rho_{\mathrm{HF}} \rangle = \left\langle \sum_{ik} \rho_{\mathrm{HF}i} V_i \lambda_k V_k \psi_k - \sum_{ik} \rho_{\mathrm{HF}k} V_k \lambda_k V_i \psi_i \right\rangle = 0 .$$

Now we take the scalar products of the first eq. (3.26) with ρ_{HF} and of the second with ψ and subtract one from the other. Then it follows from (3.32) that

$$\langle K\rho_{\mathrm{HF}} \boldsymbol{n} \rangle = K_0 \langle \psi \boldsymbol{F} \rangle . \quad (3.33)$$

As we have already shown above, it is not difficult to prove now that the difference between the rotational e.m.f. on the equilibrium and on the non-equilibrium orbits can be represented in the form

$$\frac{e}{2\pi c}\frac{\partial \Delta \Phi}{\partial t} = -\frac{\Delta E}{\omega_s^2}\left(\frac{\partial \omega}{\partial t}\right)_{E_s} . \quad (3.34)$$

From the definition of quantity $U(E, \varphi)$ we obtain, with a precision up to quadratic terms of the high frequency field,

$$\left(\frac{\partial U}{\partial E}\right)_{E_s} = \frac{2\pi}{\beta_s^2 E_s K_0^2} \left\langle \boldsymbol{\tau}\left(\psi \frac{\partial}{\partial \boldsymbol{\rho}}\right) \boldsymbol{\mathscr{E}} + \frac{\mathrm{d}\psi}{\mathrm{d}\vartheta} \boldsymbol{\mathscr{E}} \right\rangle . \tag{3.35}$$

On the other hand, according to the definition of the phase,

$$\left(\frac{\partial}{\partial \varphi}\right)_{\vartheta = \text{const.}} = \frac{1}{q\omega_s} \left(\frac{\partial}{\partial t}\right)_{\vartheta = \text{const.}} , \tag{3.36}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} \langle \psi \boldsymbol{F} \rangle = \frac{1}{q\omega_s} \left\langle \psi \frac{\partial \boldsymbol{F}}{\partial t} \right\rangle . \tag{3.37}$$

Introducing the expressions (3.24), (3.33), (3.34) and (3.37) in (3.31) we obtain

$$\omega_s \mathscr{K} \Gamma(\varphi, t) = \frac{ec}{\beta_s \omega_s} \left\langle \frac{\psi}{\omega_s} \frac{\partial \boldsymbol{\mathscr{E}}}{\partial t} + \frac{\mathrm{d}\psi}{\mathrm{d}\vartheta} \boldsymbol{\mathscr{E}} + \psi \frac{\partial \boldsymbol{\mathscr{E}}}{\partial \vartheta} \right\rangle = \frac{ec}{\beta_s E_s \omega_s K_0} \left\langle \frac{\mathrm{d}}{\mathrm{d}\vartheta} (\psi \boldsymbol{\mathscr{E}}) \right\rangle_{\rho_0 = 0} \equiv 0 . \tag{3.38}$$

We have made use here of the periodicity of the functions $\psi(\vartheta)$ and $\boldsymbol{\mathscr{E}}(\vartheta, \boldsymbol{\rho}, t)$ with respect to ϑ, and also of the relation

$$\frac{\partial \boldsymbol{\mathscr{E}}_\tau}{\partial \boldsymbol{\rho}} + \frac{1}{c} \left[\boldsymbol{\tau} \frac{\partial \boldsymbol{h}}{\partial t} \right] = K_0 \frac{\partial \boldsymbol{\mathscr{E}}_\perp}{\partial \vartheta} , \tag{3.39}$$

which may be shown to represent simply Maxwell's equation

$$\mathrm{rot}\, \boldsymbol{\mathscr{E}} = -\frac{1}{c} \frac{\partial \boldsymbol{h}}{\partial t} .$$

The only change one can achieve in the phase equation by modifying the accelerating and the magnet systems is the replacement of $V \cos \varphi$ by an arbitrary periodical function $U(\varphi)$ in the finite general case. For this it is necessary that the accelerating field should depend upon time in a non-harmonic manner, that is with radiofrequencies of different harmonics present at the same time. In all cases, dissipative terms do not exist in the phase equation, which, when the parameters are constant, describes an undamped oscillatory process. Attempts to obtain damping of the phase oscillations, for example through dependence of accelerating voltage on radius or of frequency on phase, cannot lead anywhere, as these effects are always present together and as the sum of both gives identically zero [7–9]. Let us stress that this result, as well as the general form of the phase equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{E_s \mathrm{d}\varphi/\mathrm{d}t}{\mathscr{K} \omega_s^2} \right] = \frac{qeV}{2\pi} \left[U(E_s, \varphi) - U(E_s, \varphi_s) \right] , \tag{3.40}$$

do not depend on the practical form of the accelerating system which may consist of a resonator, a drift tube, a high-frequency transformer, etc. Damping of the phase oscillations is possible only through a variation of their parameters (see § 4) or in the presence of energy losses, for instance by radiation (ch. 5).

4. SELF-FOCUSING REGIME AND PHASE OSCILLATIONS

Before examining eq. (3.23) mathematically, let us consider a mechanical analogy which is useful to clarify the mechanism of self-focusing and the processes associated with it. We have said already that when the

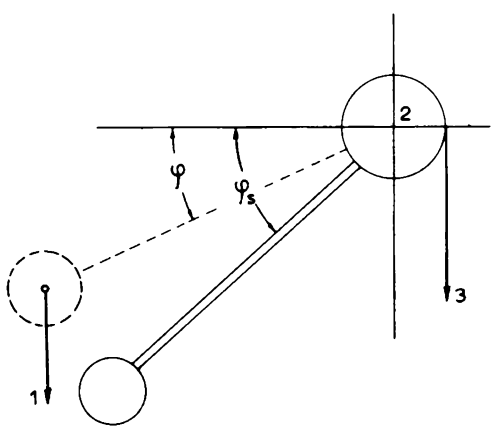

Fig. 36. Pendulum with external couple. 1 – force of gravity; 2 – axis of rotation; 3 - external force.

parameter remains constant, eq. (3.23) represents an oscillatory process taking place in a potential well of the form

$$W(\varphi) = -\frac{qeV}{2\pi}\int(\cos\varphi - \cos\varphi_s)\,d\varphi = -\frac{qeV}{2\pi}(\sin\varphi - \varphi\cos\varphi_s)\,. \quad (4.1)$$

A motion of this sort permits a simple mechanical interpretation. It is the oscillation of a pendulum on which an external couple is acting (fig. 36).

Now, if the inertial moment of the pendulum is equal to $M = E_s/\mathcal{K}\omega_s^2$,

the couple due to the force of gravity to $qeV/2\pi \cos\varphi$, and the couple due to the external force to $-qeV/2\pi \cos\varphi_s$, the equation of motion of the pendulum is exactly (3.23) even in the case where V, M and φ_s depend explicitly on time.

Let us assume for the time being that these quantities are constant and consider the possible regimes of oscillation of the pendulum. First of all, its equilibrium position is evidently $\varphi = \varphi_s$*. In our case, this state corresponds to the equilibrium motion, which is exactly in synchronism with the high-frequency field. The non-equilibrium motion is

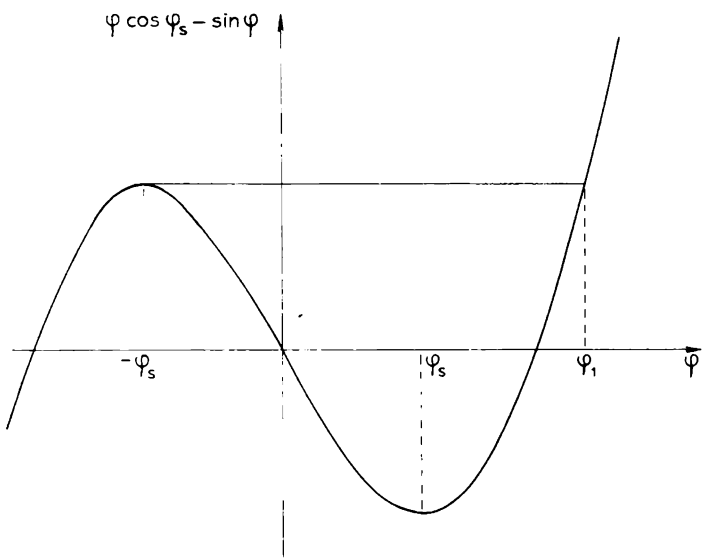

Fig. 37. Potential well $W(\varphi)$.

characterized by more or less important excursions from point $\varphi = \varphi_s$. In fig. 37 is represented the form of the potential well $W(\varphi)$ in which the motion takes place (for $M > 0$). It is easy to see that for not too large excursions away from φ_s the pendulum performs oscillations and the angle φ may vary between finite limits. In accelerators such phase

* The sign of the equilibrium phase is determined by (3.15). We shall consider in the future that φ_s has the same sign as $M = E_s/\omega_s^2 \mathscr{K}$.

oscillations are called 'synchrotron oscillations'. Correspondingly, the energy experiences some oscillations around its equilibrium value and is equal to it only in the mean. Finally, for a motion that is strongly unbalanced, there is a rotation during which the pendulum revolves constantly in one and the same direction. The corresponding particles in the accelerator fall out of synchronism with the high-frequency field and cannot be accelerated, so that the energy difference between them and the equilibrium particles increases without limit.

From what we have said is clear that for successful acceleration the energy and the phase of the particles should not differ too strongly from the equilibrium values. Let us see now how severe these requirements are. For this we shall make use of the phase plane method, which is widely applied for the investigation of linear oscillations.

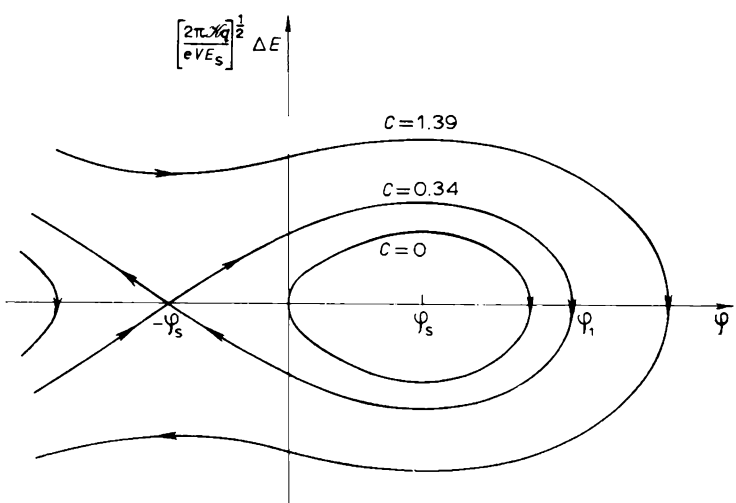

Fig. 38. Phase trajectories of synchrotron oscillations for $\cos \varphi_s = 0.5$.

The equation of phase trajectories can be obtained as the integral of the synchrotron oscillation equation. For this purpose it is sufficient to multiply both sides of (3.23) by $d\varphi/dt$ and integrate with respect to time. Taking into account relationship (3.19), we then obtain the phase trajectory equation in the form

$$\left(\frac{2\pi q\mathscr{H}}{eE_s V}\right)^{\frac{1}{2}} \Delta E = \pm[2(C+\sin\varphi-\varphi\cos\varphi_s)]^{\frac{1}{2}} . \tag{4.2}$$

The corresponding curves for various constants C are plotted in fig. 38. First of all, it is evident that all possible phase trajectories are sharply divided in two groups: the finite or closed ones, and the infinite (open) ones. The boundary between the two is the so-called 'separatrix' ('bucket limit'), the equation of which has evidently the form (4.2) with

$$C = C_c = \sin\varphi_s - \varphi_s\cos\varphi_s . \tag{4.3}$$

Corresponding to the motion along the separatrix on fig. 38 are oscillations between the points φ_1 and $-\varphi_s$, which are turnover points, that is points where $d\varphi/dt$ and ΔE tend towards zero. Starting from there,

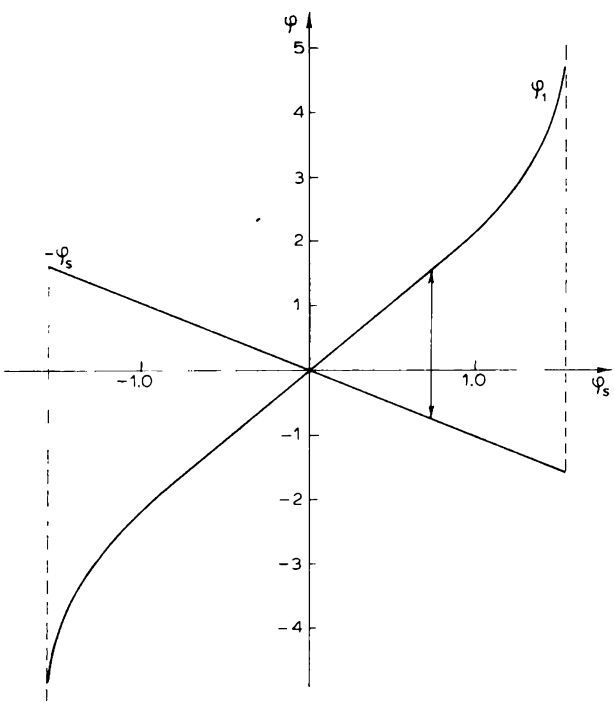

Fig. 39. Position of the turnover points as a function of equilibrium phase. The vertical arrow determines the bucket dimension for a given φ_s.

we also obtain condition (4.3). The right turnover point φ_1 satisfies evidently the equation ··

$$\sin \varphi_s + \sin \varphi_1 - (\varphi_s + \varphi_1) \cos \varphi_s = 0 , \tag{4.4}$$

the solution of which is represented on fig. 39. Thus when the equilibrium phase changes, the bucket experiences characteristic variations. For $\varphi_s = \pm \frac{1}{2}\pi$, that is in the case where the mean particle energy neither increases nor decreases, its extension in terms of φ is 2π and the form is symmetrical. For $|\cos \varphi_s| < 1$, the bucket becomes asymmetrical and no longer occupies the whole phase space (see fig. 38). This signifies that particles lying inside the separatrix form the bunch whose azimuthal dimensions do not exceed $|\varphi_1 + \varphi_s|/q$. For harmonic numbers different from unity there are simultaneously q such bunches, equally distributed along the orbit and circulating with the equilibrium frequency. Inside each of them the particles revolve around their equilibrium point.

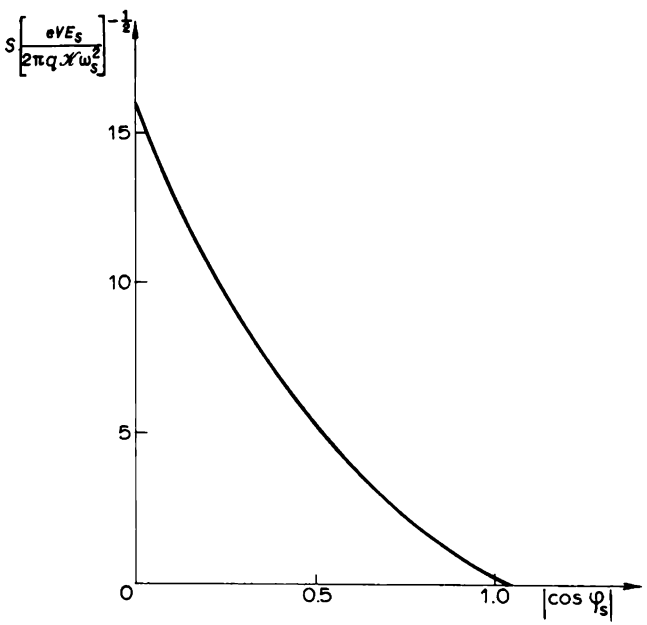

Fig. 40. Area of the stability region as a function of $\cos \varphi_s$.

From the general equation of phase trajectories, one sees easily that the particle which experiences the greatest excursion away from the equilibrium energy is the particle which moves along the separatrix and goes through the point $\varphi = \varphi_s$. This excursion is equal to

$$\Delta E_c = 2\left[\frac{eVE_s(\sin\varphi_s - \varphi_s\cos\varphi_s}{2\pi q\mathscr{H}}\right]^{\frac{1}{2}} \tag{4.5}$$

and has evidently the sense of the maximum possible energy spread of a beam accelerated under the self-focusing regime. The radial orbit excursion related to it can easily be calculated by the formula

$$x_0 = \psi(\vartheta)\frac{\Delta E_{max}}{\beta_s^2 E_s K_0}.$$

The total area contained inside the separatrix is an important feature of the accelerator, as it characterizes in an indirect manner the number of particles which may be captured into the acceleration regime. The azimuthal dimension of the bucket being proportional to q^{-1}, and the radial dimension to $q^{-\frac{1}{2}}$, the total surface decreases with growing harmonic number, as $q^{-\frac{3}{2}}$. As, however, there are q bunches simultaneously in an accelerator, the useful area of the phase space $(\Delta E/\omega_s, \vartheta)$ varies* as $q^{-\frac{1}{2}}$. In fig. 40 the dependence of this useful area on the value of the equilibrium phase is given in dimensionless co-ordinates. It is natural that this area becomes a maximum for $\varphi_s = \pm\frac{1}{2}\Pi$ and tends towards zero for $\varphi_s = 0$, the bucket shrinking to a point. If the accelerating field is small enough to make $|\cos\varphi_s| > 1$, the domain where phase trajectories are closed vanishes altogether and self-focusing becomes impossible.

Particles which lie inside the stable region of phase space move along closed phase trajectories when they accomplish synchrotron oscillations. Knowing the equations of the phase trajectories, it is not difficult to find the period of these oscillations. In fact, according to (4.2)

$$\frac{d\varphi}{dt} = \pm\left[\frac{eV\omega_s^2 q\mathscr{H}}{\pi E_s}(C + \sin\varphi - \varphi\cos\varphi_s)\right]^{\frac{1}{2}} \tag{4.6}$$

* In some cases the maximum permissible energy variation may well be limited, not by the bucket dimensions, but by the aperture of the vacuum chamber or the permissible frequency change of betatron oscillations.

and therefore the synchrotron oscillation period is

$$T_s = 2\left[\frac{qeV\omega_s^2 \mathscr{H}}{\pi E_s}\right]^{-\frac{1}{2}} \int_{\varphi_{min}}^{\varphi_{max}} [C + \sin\varphi - \varphi\cos\varphi_s]^{-\frac{1}{2}} d\varphi . \qquad (4.7)$$

It is easily seen that the synchrotron oscillation period depends in general on the phase trajectory, that is on the turnover points φ_{min} and φ_{max}, determined by the constant C

$$C = -\sin\varphi_{min} + \varphi_{min}\cos\varphi_s = -\sin\varphi_{max} + \varphi_{max}\cos\varphi_s . \qquad (4.8)$$

In particular, one should put in here $\varphi_{min} = -\varphi_s$ and $\varphi_{max} = \varphi_1$. Then it appears that at the lower limit the integral in (4.7) grows logarithmically, i.e. the period of the motion along the separatrix becomes

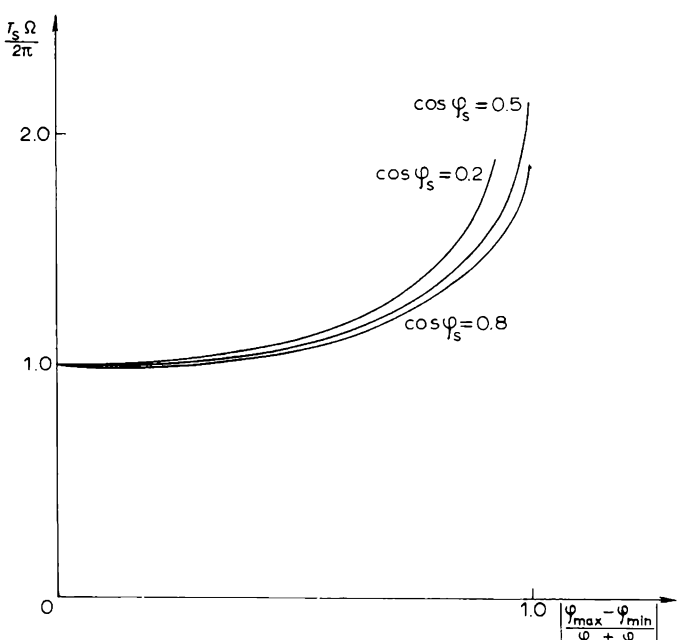

Fig. 41. Dependence of the period of synchrotron oscillations on their amplitude.

infinitely large. This is easily understood as the separatrix goes through a particular point of phase space (a saddle-point) for $\varphi = -\varphi_s$, $\Delta E = 0$, near to which the velocity of the displacement of the representative point tends toward zero. The second particular point ($\varphi = \varphi_s$, $\Delta E = 0$) is the centre around which the synchrotron oscillations take place. It is interesting to note that for $\mathscr{K} < 0$ the character of phase oscillation changes – the stable and unstable equilibrium phases exchange positions.

The dependence of the period of synchrotron oscillations on their amplitudes is represented in fig. 41. On the axis of abcissae we represent the distance between the turnover points $|\varphi_{max} - \varphi_{min}|$, divided by the full dimension of the separatrix, i.e. the dimensionless amplitude of oscillation. It is easily seen that when the argument tends towards unity, the period grows logarithmically, whereas it tends towards a constant quantity for small oscillations. The equation for small synchrotron oscillations is easily derived by introducing the deviation from the stable equilibrium phase:

$$\eta = \varphi - \varphi_s \ll 1 . \tag{4.9}$$

Then, from (3.23), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(M\dot{\eta}) + \frac{qeV}{2\pi} \sin \varphi_s \eta = 0 , \qquad M = \frac{E_s}{\omega_s^2 \mathscr{K}} , \tag{4.10}$$

which is the equation of the usual harmonic oscillator*. Hence the frequency of small synchrotron oscillations becomes

$$\Omega = \left[\frac{qeV \sin \varphi_s}{2\pi M}\right]^{\frac{1}{2}} = \omega_s \left[\frac{qeV\mathscr{K} \sin \varphi_s}{2\pi E_s}\right]^{\frac{1}{2}} \tag{4.11}$$

It is seen that the synchrotron oscillation frequency is about $\sqrt{(eV/E)}$ times smaller than the revolution frequency, so they really describe a slow average motion. To one phase oscillation correspond usually hundreds of revolutions of the particle.

Let us see now how phase trajectories behave for small changes of the motion parameters. We should stress that when deriving eqs. (3.23) we did not make any special assumptions on the time dependence of the magnetic field, radiofrequency and accelerating voltage. We only

* Strictly speaking, when the equilibrium phase depends on time, it is the term $-(\mathrm{d}/\mathrm{d}t)(M\dot{\varphi}_s)$ which should appear on the right-hand side of (4.10).

stated that they change little in a time of the order of one particle revolution. We shall now study the case when they vary, this time slowly with respect to the characteristic frequency of synchrotron oscillations. Although this condition is much more strict than the adiabaticity condition of the betatron oscillation parameters, it is usually fulfilled in practice, so that the synchrotron oscillation period turns out to be extremely small in comparison to the duration of the acceleration cycle.

As adiabatic invariant defining a variation of the parameters one can take, as we know, the area enclosed by a closed phase trajectory [10], or, in our case

$$\oint \frac{\Delta E \, \mathrm{d}\varphi}{\omega_s} = 2\left[\frac{eVE_s}{\pi q \omega_s^2 \mathscr{K}}\right]^{\frac{1}{2}} \int_{\varphi_{min}}^{\varphi_{max}} (C + \sin\varphi - \varphi\cos\varphi_s)^{\frac{1}{2}} \, \mathrm{d}\varphi = \text{const.}, \quad (4.12)$$

where C is determined from (4.8). The results of the numerical integration are shown in the graph of fig. 42, with the help of which one can obtain the adiabatic variation law of the amplitude' of phase oscillations $A = \frac{1}{2}(\varphi_{max} - \varphi_{min})$. It is still simpler to obtain the law of adiabatic damping of linear phase oscillations. In fact, in this case, our task reduces itself to the determination of the adiabatic invariant of the harmonic oscillator (see ch. 2 § 6):

$$A = \text{const.} \left(\frac{eVM \sin\varphi_s}{2\pi}\right)^{-\frac{1}{4}} = \text{const.} \left(\frac{\omega_s^2 \mathscr{K}}{E_s V \sin\varphi_s}\right)^{\frac{1}{4}}. \quad (4.13)$$

It is often of interest to know the radial dimension of the bunch or its energy spread. As follows from (4.5), the energy spread in the bucket grows with accelerating voltage (for $\varphi_s = \text{const.}$) and the equilibrium energy of the particle. As regards small synchrotron oscillations, we have for them

$$\Delta E_{max} \approx \frac{E_s}{\mathscr{K}\omega_s}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)_{max} \approx \frac{E_s \Omega A}{\mathscr{K}\omega_s} \approx \left(\frac{\omega_s^2 E_s V \sin\varphi_s}{\mathscr{K}}\right)^{\frac{1}{4}}. \quad (4.14)$$

It should be stressed that under practical conditions the bunch dimensions by no means coincide with the bucket dimensions. As small synchrotron oscillations are damped relatively quickly when the energy is

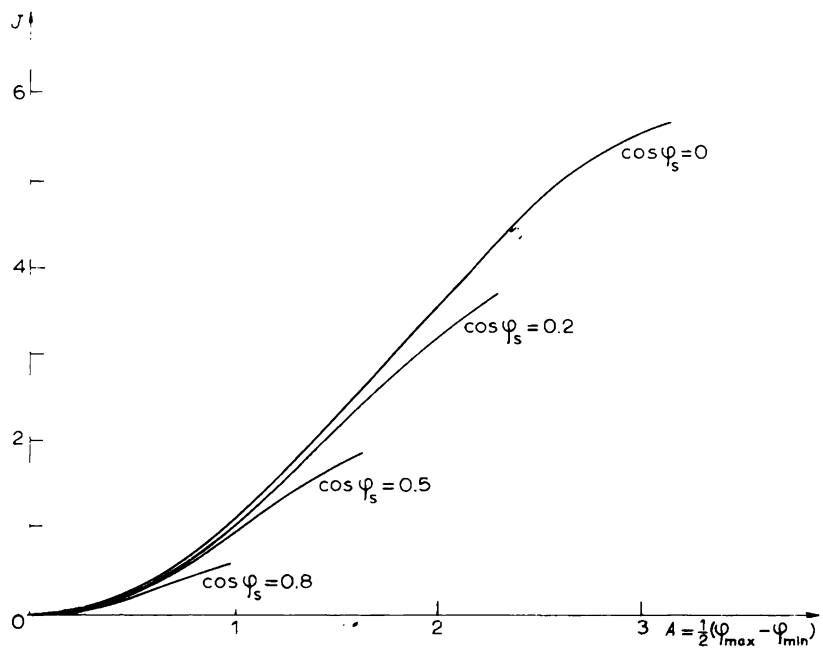

Fig. 42. For the calculation of the adiabatic invariant of synchrotron oscillations. The quantity

$$J = \int_{\varphi_{min}}^{\varphi_{max}} [2(C + \sin\varphi - \varphi\cos\varphi_s)]^{\frac{1}{2}} d\varphi$$

is represented along the ordinate axis.

raised, some time after the beginning of the acceleration most of the particles will be grouped near the equilibrium point, so that a large part of phase space inside the bucket usually remains 'free'.

Let us now consider two limiting cases: ultra-relativistic and non-relativistic energies, taking for simplicity $\varphi_s = \text{const}$.

(1) $\gamma_s \gg 1$. The particle speed is practically equal to the speed of light, and the frequency changes only because of the variation of orbit length: $\omega_s \propto \Pi_s^{-1}$. From (3.4) it is easily shown that

$$\mathscr{K} \approx \alpha , \tag{4.15}$$

so that the amplitude of small phase oscillations is proportional to

$$A_\varphi \propto (\Pi_s^2 E_s V)^{-\frac{1}{4}}, \tag{4.16}$$

and the amplitude of small radial synchrotron oscillations, that is radial oscillations of the closed orbit, to

$$A_x = \psi_{max} \frac{\Delta E_{max}}{K_0 \beta_s^2 E_s} \propto \psi_{max} \left(\frac{V}{E_s^3 \Pi_s}\right)^{\frac{1}{4}}. \tag{4.17}$$

The bucket dimension in terms of energy is proportional to

$$\Delta E_c \propto (V E_s)^{\frac{1}{2}}, \tag{4.19}$$

and its dimension along the radius to

$$x_c = \psi_{max} \frac{\Delta E_c}{K_0 \beta_s^2 E_s} \propto \psi_{max} \left(\frac{V \Pi_s}{\beta_s^2 E_s}\right)^{\frac{1}{2}} \tag{4.19}$$

In particular, in accelerators of the synchrophasotron type where the radius is practically constant, small synchrotron oscillations are damped slowly, as $E_s^{-\frac{1}{4}}$ (radial synchrotron oscillations as $E^{-\frac{1}{2}}$, that is somewhat faster than betatron oscillations). In other words, the central part of the bunch is compressed in both directions in the course of acceleration.

In fixed field accelerators the orbit grows as a rule with energy so that these effects are somewhat larger than in the synchrotron.

(2) Non-relativistic case, $\gamma_s \ll 1$. From expression (3.4) it follows that, with a precision up to terms $\propto \beta_s^4$

$$\frac{\omega_s^2}{E_s} \mathscr{K} \approx \frac{(2\pi)^2}{\Pi_s^2 m_0} (\alpha - 1) \left(1 - \frac{\beta_s^2}{2} \frac{3-\alpha}{1-\alpha}\right). \tag{4.20}$$

Thus, in the non-relativistic region phase oscillations may begin by experiencing a small increase under conditions where

$$\frac{3-\alpha}{1-\alpha} < 0 \quad \text{or} \quad 1 < \alpha < 3. \tag{4.21}$$

i.e. in a weak-focusing accelerator. In the cases interesting in practice such an amplitude increase is small and is quickly replaced by a decrease (see also ch. 7). It is of interest to note that in both relativistic and non-relativistic cases the product $(\Delta E_{max} A_\varphi)/\omega_s$ remains constant for small synchrotron oscillations for adiabatic changes of the parameters. This was evidently to be expected, as this quantity is proportional to the

area enveloped by the phase trajectory, which according to Liouville's theorem is an adiabatic invariant. These considerations, however, are not applicable to the separatrix as the area which it envelopes in the plane $(\Delta F\ \omega_s, \varphi)$ is directly proportional to $(VE_s/\omega_s^2 \mathscr{H})^{\frac{1}{2}} \neq \text{const}$. This apparent contradiction is explained by the fact that the conditions of adiabaticity are not fulfilled on the bucket boundary as the period of revolution on it is infinitely large. Thus, strictly speaking, the separatrix is only a phase trajectory for $dE_s/dt = 0$, that is for $\cos \varphi_s \approx 0$ and for $V = \text{const}$. On the contrary, the real phase trajectories differ slightly from the separatrix, especially close to the saddle-points where the adiabaticity conditions are worst fulfilled. These questions are of basic importance, for instance when studying the capture of particles into the self-focusing regime [11, 12], see also § 8.

Thus, when the self-focusing condition $|\cos \varphi_s| < 1$ is satisfied, the particles perform slow oscillations of phase and radius (or energy) around an equilibrium particle and are accelerated like it on the average. The amplitude of this oscillation must not exceed a fixed limit, otherwise the motion becomes unstable and the particles fall out of the acceleration regime. A slow variation of the particle energy and the parameters of the device lead to a variation of the amplitude of synchrotron oscillations; in particular an increase of the energy leads as a rule to a damping of the same. The action of much faster parameter changes on the synchrotron motion requires special investigation (see § 6).

5. SIMULTANEOUS CONSIDERATION OF SYNCHROTRON AND BETATRON OSCILLATIONS

We have pointed out already that, starting out from simple physical representations, it was possible to consider the motion of particles in an accelerator as the superposition of fast (betatron) oscillations, not depending in the first approximation on the variation of particle energy, and slow (synchrotron) oscillations of energy and phase around an equilibrium value. Such a picture, although based on an approximation, gives a relatively simple description of the motion which is practically always convenient for calculations and estimates.

It is important, however, to know how accurate this approximation is, that is to what degree one may consider betatron and synchrotron oscillations as independent. From what was explained above, it appears that the motion in an accelerator can be considered as a motion in an oscillatory system with three degrees of freedom: two 'fast' ones, to which correspond the partial frequencies $\nu_x\omega$ and $\nu_z\omega$ and a 'slow' one, with a partial frequency Ω

(see, for instance, [13]). As these degrees of freedom are linked, it is important to make clear by how much the normal (eigen) frequencies in such a system differ from the partial ones, that is to what extent the results of the preceding paragraph, based on the separate study of betatron and synchrotron oscillations, can be applied.

Let us not prescribe any special configuration of the high-frequency field. We shall only assume that its frequency is close to a multiple of the particle revolution frequency, so that an equilibrium orbit exists. In addition, we shall limit ourselves to the motion in the $z=0$ plane, as treatment of the three-dimensional case leads to the same results, but is somewhat more difficult.

From the general equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\boldsymbol{r}}) = \frac{e}{c}[\dot{\boldsymbol{r}}\boldsymbol{B}] + \frac{e}{c}[\dot{\boldsymbol{r}}\boldsymbol{b}] + e\boldsymbol{\mathscr{E}}, \tag{5.1}$$

where $\boldsymbol{b}$ and $\boldsymbol{\mathscr{E}}$ are high-frequency fields, small with respect to $\boldsymbol{B}$, let us calculate a similar equation for particles moving along the equilibrium orbit without betatron oscillation. We then get the following linearized vector equation

$$\Delta\dot{E}\dot{\boldsymbol{r}}_s + \Delta E\ddot{\boldsymbol{r}}_s + \dot{E}_s\dot{\boldsymbol{\rho}} + E_s\ddot{\boldsymbol{\rho}} - ec[\dot{\boldsymbol{\rho}}(\boldsymbol{B}+\boldsymbol{b})_s] - ec[\dot{\boldsymbol{r}}_s\Delta\boldsymbol{B}] = ec[\dot{\boldsymbol{r}}_s\Delta\boldsymbol{b}] + ec^2\Delta\boldsymbol{\mathscr{E}}, \tag{5.2}$$

where $\boldsymbol{\rho} = \boldsymbol{r} - \boldsymbol{r}_s$, and the sign Δ indicates deviation from the corresponding equilibrium values. In the natural co-ordinate system, related to the equilibrium orbit, one may represent the vector $\boldsymbol{\rho}$ in the form

$$\boldsymbol{\rho} = \boldsymbol{n}x + \boldsymbol{\tau}\sigma = \boldsymbol{n}x - \boldsymbol{\tau}\frac{\eta}{qK_0}, \tag{5.3}$$

where σ is the longitudinal displacement of the particle from the equilibrium position, corresponding to a phase $\eta = \varphi - \varphi_s$.

To obtain a third equation for the quantities x, σ and ΔE we shall use the formula $E = E_0(1-\dot{\boldsymbol{r}}^2/c^2)^{-\frac{1}{2}}$, with which one can represent ΔE in terms of $\dot{\boldsymbol{\rho}}$:

$$\frac{\Delta E}{E_s} = \gamma_s^2\frac{\dot{\boldsymbol{r}}_s\dot{\boldsymbol{\rho}}}{c^2}. \tag{5.4}$$

Projecting eq. (5.2) on $\boldsymbol{n}$ and $\boldsymbol{\tau}$ and changing over to ϑ as new independent variable, we obtain the following system of linear differential equations, which describe both the synchrotron and betatron motions:

$$x'' + g_x x = -\frac{(EvK_0)'}{(EvK_0)}x' + \frac{K}{K_0^2}\frac{\Delta E}{\beta^2 E} + \frac{1}{E\beta^2 K_0^2}\Delta F_n, \tag{5.5a}$$

$$\sigma' + \frac{K}{K_0}x = \frac{1}{K_0\gamma^2\beta^2}\frac{\Delta E}{E}, \tag{5.5b}$$

$$\left(\frac{\Delta E}{v}\right)' = \frac{1}{vK_0}\Delta F_\tau, \tag{5.5c}$$

where $\sigma = -\eta\, qK_0$ and F_τ, F_n are the components of the force acting on the particle due to the high-frequency field:

$$\Delta \boldsymbol{F}(\sigma, x, t) = e\beta[\boldsymbol{\tau}\, \Delta \boldsymbol{b}] + e\, \Delta \boldsymbol{\mathscr{E}} \; ; \qquad \Delta \boldsymbol{F}(0, 0, t) \equiv 0 \, . \tag{5.6}$$

To find the correction to the frequency of betatron oscillations caused by the existence of the coupling, let us use the method of averages. We shall look for a particular solution of the form

$$\begin{aligned} x &= \quad a f_x e^{i\nu\vartheta} + \text{c.c.} \\ \sigma &= -a f_\sigma e^{i\nu\vartheta} + \text{c.c.} \end{aligned} \tag{5.7}$$

Considering that in the zero approximation $\Delta E = 0$ corresponds to the betatron oscillations, we find with the help of (5.5b) the following expression for the periodic function $f_\sigma(\vartheta)$:

$$f_\sigma(\vartheta) = \frac{e^{-i\nu\vartheta}}{e^{i\nu\vartheta_0} - 1} \int_{\vartheta}^{\vartheta_0 + \vartheta} \frac{K}{K_0} f_x(\vartheta) e^{i\nu\vartheta} d\vartheta \, . \tag{5.8}$$

Comparing (5.8) with (1.3) it is easy to see that the function $f_\sigma(\vartheta)$ defined in such a manner will satisfy the following relation

$$f_x f_\sigma^* - \text{c.c.} = 2i\psi(\vartheta) \, . \tag{5.9}$$

Introducing solution (5.7) in (5.5) and varying the constants a and a^* we obtain the shortened equations

$$a' = \frac{a}{2i} \left\langle f^* \left[-\frac{(EvK_0)'}{(EvK_0)} (f_x' + i\nu f_x) + \frac{K}{K_0^2} \frac{\Delta E}{\beta^2 E} \frac{e^{-i\nu\vartheta}}{a} + \right. \right.$$

$$\left. \left. + \frac{1}{E\beta^2 K_0^2} \frac{\partial F_n}{\partial x} f_x - \frac{1}{E\beta^2 K_0^2} \frac{\partial F_n}{\partial \sigma} f_\sigma \right] \right\rangle . \tag{5.10}$$

The expression between the averaging brackets represents nothing other than twice the frequency correction $2\Delta\nu$. Considering that the law of variation of $\Delta E/E$ is determined by eq. (5.5c), one sees that the frequency correction found has the order eV/E, which is negligibly small in the cases of practical interest. This is a consequence of the fact that in the system considered, the partial frequencies ν_x and $\nu = \Omega/\omega_s$ are quite different, so that with a weak coupling they coincide practically with the normal frequencies.

Small corrections to the frequencies could play a significant role only if they were complex, that is the coupling were to cause an uninterrupted transfer of energy from one sort of oscillation to the other. However, as will be seen later, this case is never realized, in complete analogy with a system of two weakly coupled oscillators having greatly differing natural frequencies. In fact, with the help of eq. (5.5c) and relation (5.9), we get after a series of transformations

$$\text{Im}\, \Delta\nu = -\frac{1}{2} \left\langle \frac{(EvK_0)'}{(EvK_0)} + \frac{\psi}{E\beta^2 K_0^2} \left(\frac{\partial F_\tau}{\partial x} - \frac{\partial F_x}{\partial \sigma} \right) \right\rangle . \tag{5.11}$$

Starting from (5.6) one can show that the expression in curved parentheses is identical to

$$e\left(\mathrm{rot}_z\, \Delta\mathscr{E} + \frac{1}{c}\frac{\partial}{\partial t}\Delta b_z\right) \equiv 0\,. \tag{5.12}$$

As regards the first term in the averaging brackets, it is easily seen that it describes the usual adiabatic damping, as

$$EvK_0 \propto \langle B_z \rangle$$

and

$$\exp \int \mathrm{Im}\, \Delta\nu \mathrm{d}\vartheta = \frac{\text{const.}}{\langle B_z \rangle^{\frac{1}{2}}}, \tag{5.13}$$

which is exactly (6.7) of ch. 2.

So the coupling of betatron and synchrotron oscillations leads only to small purely real frequency corrections. The damping of betatron oscillations is determined by the law (6.7) of ch. 2, independent of the configuration of the high-frequency field. An analogous conclusion is true also for synchrotron oscillations, which was in fact already drawn in § 3 of this chapter. An additional damping can occur on any one of these degrees of freedom on account of some mechanism by which energy is lost, as for instance by radiation, a phenomenon which will be studied in the next chapter.

Besides the small frequency variation, the coupling leads also to fast energy oscillations. Their amplitude, however, has the relative magnitude eV/E so that one may always neglect them, exactly as in the case of the frequency corrections, and consider synchrotron and betatron oscillations as independent.

A substantial exchange of energy between betatron and synchrotron oscillations can only occur at a resonance, when the sum or difference of their frequencies is close to an integer: $\nu_x \pm \nu_s = k$. As the synchrotron oscillation frequency ν_s is small, this means that the working point must be near an integer resonance line $\nu_x \approx k$. The phenomena appearing in this case may be interpreted as follows [14]: the proximity of the integer resonance and of the parametric resonance accompanying it leads, as was shown, to slow beats of the betatron oscillation amplitude, the frequency of which may be close to the synchrotron oscillation frequency ν_s. Then a so-called 'synchrobetatron resonance' takes place, which causes a supplementary increase of the amplitude. However, the width of this resonance band is small in practice, so that the synchrobetatron resonance is not usually dangerous under practical conditions.

6. SYNCHROTRON OSCILLATIONS IN THE PRESENCE OF PERTURBATIONS

In the preceding paragraphs we have studied the self-focusing regime in ideal systems. By analogy with the betatron oscillations one can expect that in practical systems various perturbations will increase the amplitude of the synchrotron oscillations and in some cases cause particle losses [6, 15, 18]. In particular, resonance phenomena must also occur with synchrotron oscillations.

Some differences between these processes should be stressed however. If the ratio of the betatron oscillation period to the period of the external perturbation remained practically constant, the same ratio varies considerably in the case of synchrotron oscillations, whose parameters are always variable in practice. This means that resonance crossing presents a particular interest now and is, as a rule, repeated many times. In contrast, synchrotron oscillations are non-linear to a much larger degree than betatron oscillations.

We shall now neglect this last feature and consider the influence of various perturbations on small phase oscillations. Appreciation of the danger due to the different factors will have a somewhat arbitrary character; however, it will be sufficient as a rule for practical cases.

The existence of self-focusing itself is based upon the tight bond between the parameters $\partial B_z/\partial t, \mathrm{d}\omega_0/\mathrm{d}t$ and V. So it is natural that perturbations of these quantities will have a major influence on synchrotron oscillations. We shall consider below time-dependent perturbations $\Delta\omega_0, \Delta V$ and $\Delta\langle B_z\rangle$ which are the differences between the corresponding quantities and their ideal values. We note at once that one cannot consider all these quantities as independent. In particular, in synchrophasotrons, where the radiofrequency 'follows' the field, a variation of $\langle B_z\rangle$ brings about some change $\Delta\omega_0$.

When deriving the phase equation we did not impose any particular limitations on the time-dependence of the quantities ω_0, $\langle B_z\rangle_s$ and V; we assumed only that they varied slowly in comparison with the revolution frequencies. So the phase equation in its form (3.23) lends itself to the description of the perturbed as well as the unperturbed motion. It will be more convenient, however, to separate the perturbations $\Delta\omega_0$, ΔV and $\Delta\langle B_z\rangle_s$, keeping the former designations ω_0, V and $\langle B_z\rangle_s$ for the ideal values of the corresponding quantities. Then we obtain in the place of (4.10) the following linearized phase equation, which takes into account the most important perturbations:

$$\frac{1}{M}\frac{\mathrm{d}}{\mathrm{d}t}M\frac{\mathrm{d}\eta}{\mathrm{d}t} + \Omega^2\left(1 + \frac{\Delta V}{V}\right)\eta = F(t)\,. \tag{6.1}$$

Here,

$$F(t) = \Omega^2 \operatorname{ctg}\varphi_s \frac{\Delta V}{V} + \frac{1}{M}\frac{\mathrm{d}}{\mathrm{d}t}M\omega_0\left[\alpha\frac{\Delta\langle B_z\rangle}{\langle B_z\rangle} - \frac{\Delta\omega_0}{\omega_0}\right], \tag{6.2}$$

where the terms in the square brackets are not independent in general, as we remarked above.

The character of the perturbations caused by the inhomogeneous part $F(t)$ depends substantially on the rate of change of this quantity. One can divide the perturbations somewhat arbitrarily into the following three groups.

(a) Perturbations varying slowly compared with the synchrotron oscillation frequency, lead, according to what we said above, to a small variation of the position of the equilibrium point, that is of the equilibrium phase and energy.

$$\Delta\varphi_s \approx \frac{F(t)}{\Omega^2}\,; \qquad \Delta E_s = \frac{E_s}{\mathscr{K}}\left[\frac{\Delta\omega_0}{\omega_0} - \alpha\,\frac{\Delta\langle B_z\rangle_s}{\langle B_z\rangle_s}\right]. \tag{6.3}$$

We should observe that if the functional link has been realized between field and frequency, slow variations of the magnetic field may not change the equilibrium phase and energy, as the linking system automatically introduces the necessary correcting variation $\Delta\omega_0$. The question of the action exerted by the functional link $\omega_0(\langle B_z\rangle_s)$ in the case of fast perturbations is studied, for instance, in [6].

The tolerances obtained on the basis of (6.3) are relatively easy to respect, except at the critical energy ($\mathscr{K} \approx 0$) where even very small radiofrequency perturbations may lead to large excursions of the phase oscillations. We shall examine this question in more detail in § 7.

(b) Perturbations consisting of sudden discontinuities, being rapid in comparison with synchrotron oscillation frequency. The effect of these perturbations is equivalent to a sudden displacement of the equilibrium position (for ΔV) or to the appearance of an instantaneous jump $\Delta\phi$, which will excite synchrotron oscillations after rapid changes in the radiofrequency or in the field. One is interested in the increase in the amplitude of the phase oscillations after the sudden change of the parameters. The sign and value of this amplitude variation evidently depend on the phase of the oscillation at which the jump has taken place. Its value will always remain smaller than the quantity

$$\Delta A_{max} = \frac{\omega_0}{\Omega}\left|\alpha\,\frac{\Delta\langle B_z\rangle_s}{\langle B_z\rangle_s} - \frac{\Delta\omega_s}{\omega_0}\right| + \left|\frac{\Delta V}{V}\operatorname{ctg}\varphi_s\right|. \tag{6.4}$$

In this formula one must introduce the values that the parameters

had at moment of the discontinuity. The further variation of the amplitude occurs adiabatically according to the law (4.13). Expression (6.4) shows that synchrotron oscillations are most sensitive to frequency and field perturbations, and least sensitive to ΔV perturbations.

(c) The most interesting perturbation case is that where the rate of variation is comparable to the synchrotron oscillation frequency. It is precisely under such conditions that various resonance phenomena can occur, which one should study in some detail.

Let us consider first the effect of a harmonic perturbation of frequency Ω' and phase χ

$$F(t) = F_{\Omega} \sin(\Omega' t + \chi), \tag{6.5}$$

which excites resonance for $\Omega \approx \Omega'$. We have already mentioned that the synchrotron motion was characterized by a slow variation of the frequency, so that a passage through resonance in fact occurs.

Knowing that the solution of the adiabatic approximation has the form

$$\begin{aligned} \eta &= a(t) C \exp\left[\mathrm{i} \int \Omega \mathrm{d}t\right] + \text{c.c.}, \\ \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \mathrm{i} a(t) \Omega(t) C \exp\left[\mathrm{i} \int \Omega \mathrm{d}t\right] + \text{c.c.}, \end{aligned} \tag{6.6}$$

where the complex amplitude a varies according to the adiabatic law, we find by the method of variation of the constants C and C^* the following equation for the growth of the amplitude

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{F_{\Omega'} \sin(\Omega' t + \chi)}{2\mathrm{i} a(t) \Omega(t)} \exp\left[-\mathrm{i} \int \Omega \mathrm{d}t\right], \tag{6.7}$$

where $a(t)$ refers as before to the amplitude in the absence of resonance. The frequency Ω in the exponential can be expanded in a series of powers

$$\Omega = \Omega' + \left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)' \tau + \ldots, \tag{6.8}$$

where τ is the time counted from the instant of the exact resonance. Introducing this expression in (6.7) and keeping only the resonance term which gives a systematic variation of amplitude, we find

$$\Delta a = a(t)(C-1) = -a(t) \int_{-\infty}^{t} \frac{F_{\Omega'} \exp\left[-\mathrm{i}(\mathrm{d}\Omega/\mathrm{d}t)\frac{1}{2}\tau^2 + \mathrm{i}\chi\right] \mathrm{d}\tau}{4 a(\tau) \Omega(\tau)}. \tag{6.9}$$

In most cases of practical interest, one may neglect the variation of $F_{\Omega'}, \chi, a(t)$ and $\Omega(t)$ for the time during which one passes through resonance. Then the integral in (6.9) becomes a Fresnel integral. We are interested, however, in the increase in amplitude after the resonance has been passed, that is for $t \to \infty$. Then we get from (6.9)

$$\Delta a \approx -\frac{a(t)}{4a'\Omega'}\left[\frac{\pi}{|\mathrm{d}\Omega/\mathrm{d}t|'}\right]^{\frac{1}{2}} \exp\left[-\mathrm{i}(\chi \pm \tfrac{1}{4}\pi)\right], \tag{6.10}$$

where the quantities with a prime refer to the moment of the resonance, while the upper or lower sign in the exponential is to be taken according to the sign of $\mathrm{d}\Omega/\mathrm{d}t$.

From (6.10) it follows that the amplitude variation of the synchrotron oscillation depends on the phase which this oscillation displays relative to the external perturbation in resonance. As the particles may be considered as uniformly distributed with respect to the phase of the synchrotron oscillations, one can calculate the mean and the mean square amplitude variation when crossing one resonance. For this let us observe that according to (6.10), we have

$$\overline{\Delta a} = \overline{\Delta a^*} = 0\ ; \qquad \overline{(\Delta a)^2} = \overline{(\Delta a^*)^2} = 0\,, \tag{6.11}$$

where the bar indicates average over χ. So, with an accuracy up to terms quadratic in Δa, we find with the identity $A = 2|a|$

$$\overline{\Delta A} = \tfrac{1}{4}A\frac{\overline{|\Delta a|^2}}{|a|^2}, \qquad \overline{(\Delta A)^2} = \tfrac{1}{2}A^2\frac{\overline{|\Delta a|^2}}{|a|^2} = 2A\overline{\Delta A}\,. \tag{6.12}$$

These relations are often encountered in the theory of the various stochastic processes occurring in an accelerator, as for instance when studying multiple scattering on the residual gas, fluctuations due to radiative processes, etc*. (see also ch. 6).

Taking the average with respect to χ and making use of (6.12), we obtain finally

$$\begin{aligned} \overline{\Delta A} &= \frac{A(t)\pi F_{\Omega'}^2}{16\Omega'^2 A'^2 |\mathrm{d}\Omega/\mathrm{d}t|'}\,, \\ \overline{(\Delta A)^2} &= 2A\overline{\Delta A}\,. \end{aligned} \tag{6.13}$$

* It should be noted that formulae (6.12) were obtained under the assumption that the amplitude increase Δa was much smaller than the amplitude itself.

It is interesting to note that the mean amplitude increase is always positive, in spite of the fact that the amplitude can with equal probability grow or decrease when going through resonance.

Since for multiple traversals of resonance the amplitude increases can be taken as statistically independent, the mean amplitude and the dispersion of the amplitude distribution must grow in a monotonic manner when the number of traversals is increased. This is to some extent analogous to a diffusion process.

Resonance perturbations may show up in practical machines for various reasons. As an example, in the JINS synchrophasotron, because of some peculiarities of the power supply of the electromagnet, the magnetic field displays small harmonic ripples with frequencies of 600, 1200 c s etc. In this case it seemed appropriate to programme $V(t)$ in such a way as to avoid some of these resonances.

Let us now assume a somewhat different type of perturbation which is characteristic of synchrotron oscillations. The instability inherent in the magnet power supply and the high-frequency system leads to the appearance of perturbations $\Delta\omega_0$, ΔV, and $\Delta\langle B_z\rangle_s$ which are random functions of time (noise). In their spectrum there is always a frequency which resonates at a given moment with the synchrotron oscillations. Although the various components act in an incoherent manner, their resulting action in the course of the acceleration turns out to be important.

It is physically quite clear that the action of noise on the system considered can only be described statistically. To this end we need to know some statistical properties of the noise in question. An important feature of a random perturbation $F(t)$ is its spectral intensity $\Phi(\Omega)$, by which we understand the mean probable power of the process $F(t)$ per unit frequency interval.

Another important feature of a random function $F(t)$ is the 'correlation function' $\Psi(\tau)$, defined by the relation

$$\Psi(\tau) = \widetilde{F(t)F(t-\tau)} \tag{6.14}$$

(with the wavy line we shall indicate a time average, whereas the straight line will mean an average in the statistic sense, that is over the whole process). For the stationary processes which we shall consider, the correlation function does not depend explicitly upon time.

In the theory of random processes, one sees [16] that $\Psi(\tau)$ and $\Phi(\Omega)$ are related by a reciprocal Fourier transformation

$$\Phi(\Omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \Psi(\tau) e^{i\Omega\tau} d\tau ,$$
$$\Psi(\tau) = \frac{1}{2} \int_{-\infty}^{+\infty} \Phi(\Omega) e^{-i\Omega\tau} d\Omega . \tag{6.15}$$

In particular, it follows from the second relation that

$$\Psi(0) = \widetilde{\widetilde{F^2}(t)} = \int_0^{\infty} \Phi(\Omega) d\Omega , \tag{6.16}$$

This shows clearly the sense of $\Phi(\Omega)$, which is a mean power per unit frequency interval.

Let us now consider a random excitation of synchrotron oscillations, taking their adiabatic damping into account. By analogy with (6.7) we obtain the following law of variation for the complex amplitudes

$$\frac{dC}{dt} = \frac{F(t) \exp[-i \int \Omega dt]}{2ia(t)\Omega(t)} . \tag{6.17}$$

If we average this expression over the whole system, that is over the possible ways to realize the process $F(t)$, we obtain, as we could have expected, $\overline{dC}/dt = 0$. We also find

$$\frac{\overline{d|C|^2}}{dt} = \int_0^t \frac{\Psi \exp[-i \int_{t-\tau}^{t} \Omega dt] d\tau}{4a(t)a^*(t-\tau)\Omega(t)\Omega(t-\tau)} + \text{c.c.} ; \tag{6.18}$$

where we have made use of the so-called ergodic property of the stationary random processes we examine. This property resides in the fact that the average over the whole is equal to the time-average for any realization. In the cases of practical interest, the correlation function falls rapidly with increasing τ. In the interval where $\Psi(\tau)$ is different

from zero (that is in the interval of the correlation), one may neglect the adiabatic variation of frequency and amplitude. Then from (6.18) we obtain, with the help of (6.15), that for sufficiently large t (much larger than the correlation interval)

$$\overline{\frac{\mathrm{d}|C|^2}{\mathrm{d}t}} \approx \frac{\pi}{4|a|^2} \frac{\Phi(\Omega)}{\Omega^2}. \tag{6.19}$$

Thus, in the presence of noise and adiabatic damping, the amplitude varies according to the law

$$\overline{A^2_{\text{perturb}}}(t) \approx A^2(t) \left[1 + \pi \int_0^t \frac{\Phi(\Omega)}{A^2(t)\Omega^2} \, \mathrm{d}t\right], \tag{6.20}$$

where one should understand by $A(t)$ the amplitude in the absence of perturbation, that is the amplitude which varies according to the adiabatic law. In particular, neglecting adiabatic damping and initial amplitude, we arrive at the result in [15]:

$$\overline{A^2_{\text{perturb}}}(t) = \pi \int_0^t \frac{\Phi(\Omega)}{\Omega^2} \, \mathrm{d}t \,. \tag{6.20a}$$

Expression (6.20) makes it possible to calculate the permissible spectral noise density in the whole frequency band in which synchrotron oscillation can vary. To proceed from the tolerance on $F(t)$ to the tolerances on the particular forms of perturbations, one has only to keep in mind that the spectral density of quantity $BF(t)$, where B is a constant, is $B^2 \Psi(\Omega)$ and that the spectral density of the derivative $\mathrm{d}F/\mathrm{d}t$ is $\Omega^2 \Phi(\Omega)$.

Making use of these results it is easy to determine, in particular, that in the adiabatic region $\Phi(\Omega)$ depends as follows on the spectral densities of the dimensionless perturbations $\Delta\omega_0/\omega_0$, $\Delta V/V$ and $\Delta\langle B_z\rangle/\langle B_z\rangle$

$$\frac{\Phi(\Omega)}{\Omega^2} = \begin{cases} \Omega^2 \operatorname{ctg}^2 \varphi_s \, \Phi_V(\Omega) \,, \\ \omega_0^2 \alpha^2 \Phi_B(\Omega) \,, \\ \omega_0^2 \Phi_{\omega_0}(\Omega) \,. \end{cases} \tag{6.21}$$

As in the case of resonance perturbations the largest effect on syn-

chrotron oscillations is exerted by radiofrequency and field perturbations, and the smallest by the noise of the accelerating voltage. It should be mentioned that the tolerance on the noise of the magnetic field depends essentially on whether the radiofrequency is tuned automatically to the field or not. If one uses such a system, one must consider instead of the perturbation $\alpha \Delta\langle B_z\rangle/\langle B_z\rangle$, the quantity

$$\left[\alpha \frac{\Delta\langle B_z\rangle}{\langle B_z\rangle} - \frac{\Delta\omega_0}{\omega_0}\right],$$

which is due to the imperfections of the automatic tuning system.

Under certain conditions high harmonics of the mains frequency (50 c/s), which are usually present in the magnet supply circuits, can act in the same way as a random perturbation. Let us assume that the amplitudes of these harmonics are, for example, equal in the band $\Delta\Omega$ in which the synchrotron oscillation frequency varies, and that their phases are randomly distributed. Then one can consider the perturbation they create as a noise with the spectral density

$$\Phi(\Omega) = \Omega^2 \omega_0^2 \alpha^2 \left(\overline{\frac{\Delta\langle B_z\rangle}{\langle B_z\rangle}}\right)^2 \frac{1}{\Delta\Omega}.$$

The mean square value of the amplitude of the forced phase oscillations will be equal to

$$\overline{A^2_{\text{perturb}}} = \frac{\pi\alpha^2}{\Delta\Omega} \left(\overline{\frac{\Delta\langle B_z\rangle}{\langle B_z\rangle}}\right)^2 A^2(t) \int_0^t \frac{\omega_0^2}{A^2(t)}\, \mathrm{d}t .$$

If the equilibrium energy increases linearly with time, and if one can neglect the variation of ω_0 and V during the major part of the acceleration cycle, then $A(t) \propto t^{-\frac{1}{4}}$ (see 4.16) and

$$\overline{A^2_{\text{perturb}}} \approx \frac{2\pi\alpha^2}{3\Delta\Omega} \left(\overline{\frac{\Delta\langle B_z\rangle}{\langle B_z\rangle}}\right)^2 \omega_0^2 t .$$

With $\omega_0 = 10^7$ c/s, $\Delta\Omega = 10^{-2}\omega_0$, $\alpha = 3$, $t = 1$ sec, and a permissible amplitude of the perturbed oscillation of 1 rad, we obtain the condition for the mean square value of the perturbation in the interval $\Delta\Omega$:

$$\left(\frac{\Delta\langle B_z\rangle}{\langle B_z\rangle}\right)_{\substack{\text{mean square}\\ \text{perturb}}} \lesssim 10^{-5}.$$

In strong focusing magnets ($\alpha \ll 1$) this tolerance is somewhat less stringent.

Up to now we have considered only the perturbation $F(t)$ in eq. (6.1), neglecting tacitly the term $\Omega^2(\Delta V/V)\eta$. For small oscillations this term is in fact small in comparison with all other terms, but by analogy with betatron oscillation one may expect that it will excite a parametric or half-integer resonance.

One can study this problem quantitatively in exactly the same fashion as was done in ch. 3, § 8. If the perturbation $\Delta V/V$ has the form

$$\frac{\Delta V}{V} = v \sin \Omega' t\,, \tag{6.22}$$

then the exact resonance occurs when the perturbation frequency is just equal to twice the synchrotron oscillation frequency $\Omega' = 2\Omega$. Considering a fast crossing of this resonance, we obtain the following expression for the maximum value of $|C|$ after the traversal

$$|C| = \exp\left[\tfrac{1}{8}v\Omega'\sqrt{\frac{\pi}{|\mathrm{d}\Omega/\mathrm{d}t|'}}\right], \tag{6.23}$$

where primes indicate the values of the various quantities at resonance (the initial value is $|C| = 1$).

It is interesting to consider the parametric resonance of synchrotron oscillations with noise. To this end one may use formula (6.19) directly but understand for $\Phi(\Omega)$ a spectral density of value

$$\Omega^2(\Delta V/V)[a(t)C\,\mathrm{e}^{\mathrm{i}\int\Omega\,\mathrm{d}t} + \text{c.c.}].$$

In this case

$$\Phi(\Omega) \approx \Omega^4 |a(t)|'^2 |C|^2 \Phi_V(2\Omega)\,, \tag{6.24}$$

so that

$$\overline{|C|^2} = \exp\int_0^t \tfrac{1}{4}\pi\Omega^2 \Phi_V(2\Omega)\mathrm{d}t\,. \tag{6.25}$$

It is easily seen that the perturbing effect of noise is overcome in this case by the adiabatic damping of the oscillation if

$$\left|\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t}\right|_{\mathrm{ad}} > \tfrac{1}{8}\pi\Omega^2\Phi_\nu(2\Omega)\,. \tag{6.26}$$

We have already remarked that tolerances computed on the basis of our philosophy had a somewhat questionable character. This is because when one makes such estimates, one is obliged to prescribe a maximum amplitude of oscillation (usually of the order of one radian) for each type of perturbation. Actually particle losses occur when the amplitude is comparable to the bucket dimensions, so that non-linear effects become appreciable. Besides, it is insufficient for a correct tolerance calculation to find the mean square value of the amplitude, as one should really compute the particle loss due to each of the perturbations separately. Some considerations on these questions can be found in [17–19], and also in ch. 6.

In the chapter on betatron oscillations it was shown that when crossing a resonance non-linearities either aggravate its action or weaken it, depending from which side the resonance is approached. Synchrotron oscillations happen to be 'soft', as one says when the frequency decreases when the amplitude grows. The frequency of synchrotron oscillations now decreases as a rule in the course of the acceleration cycle, as it is proportional to $E_s^{-\frac{1}{2}}$. This means that non-linear effects must help when crossing resonances, i.e. diminish the harmful action of the various resonances. From this point of view, the estimates of the effect of perturbations made above on the basis of the linear theory appear somewhat exaggerated. Also, one should stress that if the mean square of the amplitude is small in comparison with the bucket dimension this does not guarantee complete absence of particle losses. An example of such statistical losses would be, for instance, particle losses due to the quantum fluctuation of radiation, which will be considered in ch. 6.

7. CRITICAL (TRANSITION) ENERGY

As was shown in previous paragraphs (see also ch. 1 § 4) the self-focusing properties of an accelerator are characterized to a large extent by the

parameter $\mathcal{K}$, which is the logarithmic derivative of the frequency with respect to energy, taken with a magnetic field constant in time.

$$\mathcal{K} = -\left(\frac{\partial \ln \omega}{\partial \ln E}\right)_{s,B} = \frac{\omega_s}{E_s}\frac{\alpha\gamma_s^2 - 1}{\gamma_s^2 - 1}. \tag{7.1}$$

The smaller the quantity $\mathcal{K}$, the larger the 'effective mass' M entering into the phase equation (3.23), and the lower the frequency of small synchrotron oscillations, i.e. the phase stability.

We have stated already that self-focusing is possible with either positive or negative $\mathcal{K}$; in the first case the stable equilibrium phase is larger than zero, in the second case, smaller than zero. In acceleration with azimuthally symmetrical fields (and in general when $\alpha > 1$) the first case is realized, as one can easily determine. In strong-focusing accelerators (and in general when $\alpha < 1$) the quantity $\mathcal{K}$ is negative for small energies. Then there exists an energy $E_s = E_k$, for which the quantity $\mathcal{K}$ tends towards zero. With a further increase in energy, this quantity becomes positive. The energy E_k is called the 'critical' or 'transition' energy, and is determined by the condition [20]:

$$\gamma_k = \alpha^{-\frac{1}{2}}. \tag{7.2}$$

The significance of this name comes from the fact that at the critical point phase stability vanishes, the phase oscillation frequency tends towards zero and self-focusing disappears.

As when one approaches the critical point the 'effective mass' $M = E_s/\omega_s^2 \mathcal{K}$ grows in absolute value, phase oscillations are strongly damped and their frequency Ω decreases constantly, so that in the immediate neighbourhood of E_k the adiabatic approximation is not applicable (see below). The quantity $M(t)$ goes suddenly from $-\infty$ to $+\infty$ when passing through E_k, and decreases quickly further down to a certain value, after which it begins to grow again slowly. Correspondingly, the phase oscillation amplitudes of those particles which one succeeds in transferring through the critical region begin to increase and then again decrease slowly.

From what has been said, one may conclude that, in the absence of special measures (even when there are no perturbations), the majority of the particles will be lost when passing through the critical point. At the moment when E_k is reached, the region of the phase oscillations for

all particles lies around the stable equilibrium phase $\varphi_s < 0$. At the instant of transition through E_k the stable phase becomes positive, whereas the particles remain grouped around $-|\varphi_s|$. The particles for which $\varphi(t_k) < -|\varphi_s|$ then lie in an unstable region (outside the bucket) and fall out of synchronism. The phases of the remaining particles begin to perform oscillations with an amplitude of the order of $2|\varphi_s|$, which corresponds to the boundary of the stability region. The sharp reduction of the effective mass $M(t)$ after the passage through the critical point leads to large swings of the phase oscillations and, as computations show the particles cross the limits of the stability region in the course of about one revolution around the equilibrium position in phase space [21].

To be able to bring the particles actually across critical energy, one must at $E_s = E_k$ change the phase of the accelerating voltage by $2|\varphi_s|$ [16, 21, 22]. The particle whose phase has been altered by $2|\varphi_s|$ are then grouped around the new equilibrium position and the accelerating regime is not perturbed. There are also other methods of passing through the critical energy [21, 23].

The problem of the critical energy appears in fact only in strong-focusing accelerators of heavy particles. The injection there is performed at non-relativistic energies, that is, at small γ, much smaller than γ_k. Thus, the CERN proton-synchrotron has a critical energy of about 5.7 GeV, whereas the protons are injected at a kinetic energy of 50 MeV. In electron accelerators, the same γ_k corresponds to an energy which is two thousand times less. Clearly one may inject in this case at an energy which is larger than E_k, and in this way eliminate the difficulties due to the critical energy.

Although the question of the critical energy has a rather particular character, it is appropriate to dwell on it in some detail in view of its importance for the construction of large proton synchrophasotrons. Near transition energy it is impossible, of course, to use the adiabatic approximation, as the phase oscillation frequency tends towards zero. One may, however, with a sufficient degree of accuracy consider the quantities E_s and ω_s as constants, and approximate the function $\mathscr{K}(t)$ by a linear law.

$$\mathscr{K}(t) \approx \left.\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t}\right|_k (t - t_k) = \frac{\mathscr{K}_1}{E_k} \left.\frac{\mathrm{d}E_s}{\mathrm{d}t}\right|_k \tau\,, \tag{7.3}$$

where $\tau = t - t_k$, the subscript k indicating quantities related to the critical point, and where

$$\mathcal{K}_1 = -\frac{E_k^2}{2\omega_k}\left(\frac{\partial^2\omega}{\partial E^2}\right)_k = \frac{\alpha}{1-\alpha}. \tag{7.4}$$

As the 'effective mass' grows when approaching the critical energy, the phase oscillations must be strongly damped, a fact which is already shown by the adiabatic approximation. For this reason one may use the linearized phase equation in the form (4.10). We shall see later how far this assumption is correct.

Introducing (7.3) in (4.10) we obtain for the excursion from the stable equilibrium phase $\eta = \varphi - \varphi_s$ following equation

$$\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{\tau}\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right] + \frac{\tau qeV\omega_s^2\mathcal{K}_1(\mathrm{d}E_s/\mathrm{d}t)\sin\varphi_s}{2\pi E_s^2}\eta = -\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{\tau}\frac{\mathrm{d}\varphi_s}{\mathrm{d}\tau}\right]. \tag{7.5}$$

If $\cos\varphi_s \approx$ const. the derivative $\mathrm{d}\varphi_s/\mathrm{d}t$ differs from zero only at the point $\tau = 0$, where the equilibrium stable phase changes sign

$$\frac{\mathrm{d}\varphi_s}{\mathrm{d}\tau} = 2|\varphi_s|\,\delta(\tau). \tag{7.6}$$

The general solution of the homogeneous equation (7.5) can be expressed in terms of Bessel functions of order $\pm\frac{2}{3}$:

$$\eta = w^{\frac{1}{2}}\left[C_1 J_{\frac{2}{3}}(w) + C_2 J_{-\frac{2}{3}}(w)\right], \tag{7.7}$$

where

$$\begin{aligned} w &= \int_0^\tau \Omega\,\mathrm{d}\tau = (\Omega_0\tau)^{\frac{3}{2}}, \\ \Omega_0 &= \left[\frac{2qeV\omega_s^2\mathcal{K}_1(\mathrm{d}E_s/\mathrm{d}t)\sin\varphi_s}{9\pi E_s^2}\right]^{\frac{1}{3}}. \end{aligned} \tag{7.8}$$

Let us note that the quantity ω, defined by the relation (7.8), is always positive, so that $\sin\varphi_s$ and therefore Ω_0 change sign at the same time as τ. The constants C_1 and C_2 are determined by the initial conditions and amount, as a rule, to some tenths of radians. This corresponds to the fact that up to the moment when the critical point is reached, the

adiabatic damping has time to have some effect on the phase oscillations. Making use of the asymptotic expansions of the Bessel function one easily finds that far from the critical point ($w \gg 1$), expression (7.7) simplifies to the solution of the adiabatic approximation

$$\eta \approx w^{\frac{1}{4}} \sqrt{(2/\pi)} \left[C_1 \sin\left(\int_0^\tau \Omega \, dt - \tfrac{1}{12}\pi\right) + C_2 \cos\left(\int_0^\tau \Omega \, d\tau + \tfrac{1}{12}\pi\right)\right], \quad (7.9)$$

where the factor $w^{\frac{1}{4}}$, according to the assumptions made, can be brought into the more convenient form

$$w^{\frac{1}{4}} = (\Omega_0 \tau)^{\frac{1}{4}} = \left[\frac{\Omega_0 \mathcal{K}(\tau) 2\pi E_s}{\mathcal{K}_1 eV \omega_s \cos \varphi_s}\right]^{\frac{1}{4}}. \quad (7.10)$$

It is easily seen that the coefficient $\mathcal{K}^{\frac{1}{4}}(\tau)$ describes the adiabatic damping due to the variation of $\mathcal{K}(t)$ correctly (compare with eq. 4.13).

According to formula (7.7), small phase oscillations are strongly damped when approaching the critical energy, at the beginning according to an adiabatic law (4.13 or 7.10), and later (for $w \lesssim 1$) somewhat more slowly. Thus, the linearization of the original equation in η appears valid up to the critical point.

To investigate the behaviour of the energy in the critical region, let us use formulae (7.7) and (3.22), from which follows

$$\frac{\Delta E}{E_s} = \frac{1}{q\omega_s \mathcal{K}} \frac{d\eta}{d\tau} = \frac{\Omega}{q\omega_s \mathcal{K}} \frac{d\eta}{dw} = \frac{\Omega}{q\omega_s \mathcal{K}} w^{\frac{1}{4}} \left[C_1 J_{-\frac{1}{3}}(w) - C_2 J_{\frac{1}{3}}(w)\right] \quad (7.11)$$

One finds easily that for $w \gg 1$ this expression also simplifies to the solution of the adiabatic approximation. Thus in the critical region the energy excursion from the equilibrium value increases, reaching its maximum for $\tau = 0$. The value of this maximum may easily be found from (7.11), using the asymptotic expression for Bessel functions for small arguments. We obtain then

$$\left(\frac{\Delta E}{E_s}\right)_{\tau \to 0} \to C_1 \left[\frac{\operatorname{tg} \varphi_s \Omega_0}{q \mathcal{K}_1 \omega_s}\right]^{\frac{1}{4}} \frac{2^{\frac{1}{3}}}{\Gamma(\frac{2}{3})}. \quad (7.12)$$

The phase excursion at the critical point tends also towards a defined value

$$(\eta)_{\tau \to 0} \to C_2 \frac{2^{\frac{1}{3}}}{\Gamma(\frac{1}{3})}. \quad (7.13)$$

The behaviour of the maximum excursions of energy and phase from the equilibrium values are shown on fig. 43 (it is assumed that one has secured the passage of the particles through the critical point).

Computation shows that the maximum energy excursion from the equilibrium value may exceed several times the amplitude of the energy

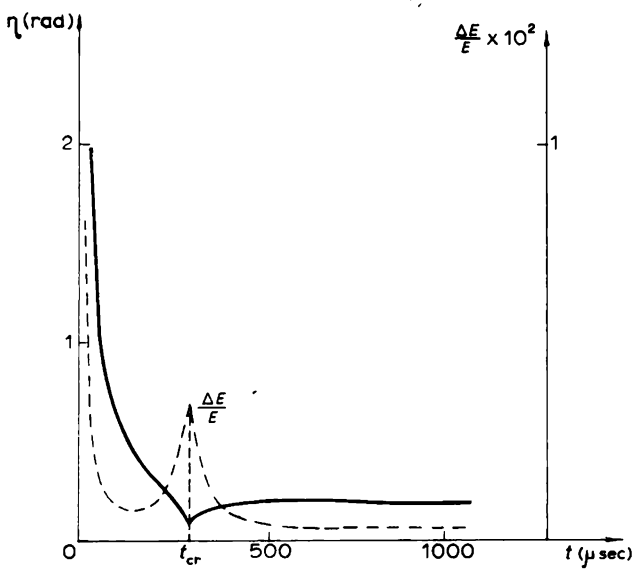

Fig. 43. Example of the amplitudes of phase and energy oscillations near the critical point t_{cr}.

oscillation in the adiabatic region. It should, however, be noted that in many cases of practical interest the relative energy spread in the beam is much smaller than at injection, since at the moment when the critical energy is reached the adiabatic damping succeeds in acting strongly on the beam. So the increase in energy spread at the critical point may in itself not present a danger.

Let us now turn to the question of whether the expressions we have derived can be applied. It is easily seen that they have a physical meaning, first when the phase focusing condition $|\cos \varphi_s| < 1$ is fulfilled, and secondly if the passage through the critical energy is carried out quickly enough, that is, if the quantity $|\cos \varphi_s|$ is sufficiently large. On the other hand the limit $(\Delta E/E_s)_{\tau=0}$ increases because of the factor $\mathrm{tg}^{\frac{1}{2}} \varphi_s$, and may no longer be considered

as a small quantity, an assumption which was essential for our derivation of the phase equation.

For these reasons it is necessary, strictly speaking, to keep the quadratic term in the expansion

$$\frac{d\varphi}{d\tau} = \mathcal{K}(\tau)\omega_s\left(\frac{\Delta E}{E_s}\right) + \mathcal{K}_1\omega_s\left(\frac{\Delta E}{E_s}\right)^2 + \ldots \tag{7.14}$$

as this term turns out to be equal to the linear one in the critical region. Clearly, one can neglect its effect only if the passage through the critical point is performed quickly enough. To obtain an estimate, one can apply the method of variation of constants to C_1 and C_2, and require that their variation during the time the particles remain in the critical region should be small. The criterion one obtains has the form

$$\frac{\operatorname{tg}\varphi_s}{q}\, C_{1,2}| \ll 1\,. \tag{7.15}$$

which confirms the statements made above: the passage through the critical energy should happen quickly enough and the phase oscillations during the passage over the critical point should be sufficiently small.

As regards phase oscillations of large amplitude, the mathematical difficulties are such that one can examine them only qualitatively, in the case where the parameter $\mathcal{K}(\tau)$ varies very slowly and adiabatically (with a very small $|\cos\varphi_s|$, see [21, 23]).

Above all, it should be observed that the equations of the synchronous motion in the variables φ and $\Delta E/\omega_s$ may be derived from the Hamiltonian

$$\mathcal{H} = -\frac{eV}{2\pi}(\sin\varphi - \varphi\cos\varphi_s) + \frac{q\omega_s^2\mathcal{K}}{E_s}\frac{1}{2}\left(\frac{\Delta E}{\omega_s}\right)^2 + \frac{q\omega_s^3\mathcal{K}_1}{3E_s^2}\left(\frac{\Delta E}{\omega_s}\right)^3 + \ldots, \tag{7.16}$$

where one may neglect terms of higher order in the canonical momentum $\Delta E\ \omega_s$ in the approximation in which we are interested. If $\mathcal{K}$ varies adiabatically, the phase trajectories must lie near the curves $\mathcal{H} = \text{const}$. the equation of which, in the dimensionless coordinates

$$u = \frac{\Delta E}{E_s}\left[\frac{\pi q\mathcal{K}_1 E_s}{3eV}\right]^{\frac{1}{3}}\ ;\ \varphi \tag{7.17}$$

is

$$u^3 + \varepsilon u^2 - (\sin\varphi - \varphi\cos\varphi_s) = C = \text{const.} \tag{7.18}$$

The parameter ε defines the proximity within which one is from critical energy:

$$\varepsilon = \frac{\mathcal{K}}{\mathcal{K}_1}\left[\frac{9\pi q\mathcal{K}_1 E_s}{4eV}\right]^{\frac{1}{3}} \approx \left[\frac{9\pi q\alpha}{4eVE_k^2(1-\alpha)}\right]^{\frac{1}{3}}(E_s - E_k)\,, \tag{7.19}$$

where the second part of the equation is correct in the immediate vicinity of the critical point.

If we can neglect the term proportional to u^3 in the relation (7.18), we will derive from the latter the usual equation of the phase trajectories, as can be easily shown. So, in a

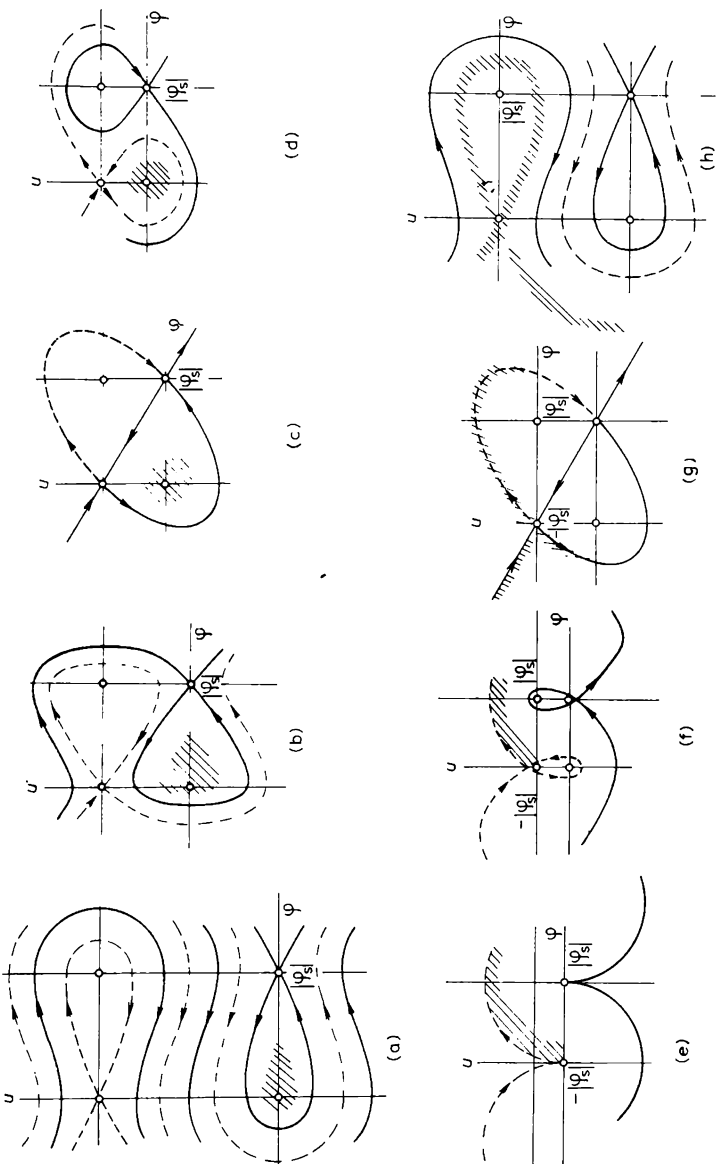

Fig. 44. Schema of the transformation of the phase trajectories in the vicinity of the critical energy. The region occupied by the beam is shaded. The dotted and full lines represent the upper and lower buckets respectively.

rather questionable manner one may take as a criterion of critical conditions the inequality $\varepsilon \lesssim 1$. The value $\varepsilon = 0$ corresponds to the critical point itself.

A schematic picture of the transformations of the curves $\mathcal{H} = \text{const.}$ when passing through the critical energy is given in fig. 44. An interesting feature of phase space in this case is the existence of a second stable point, lying at $+|\varphi_s|$, and of a second bucket. As the equilibrium energy approaches critical energy the two buckets shown in fig. 44a come near to one another, and their radial (energetic) dimensions increase. At a certain moment the buckets come into contact, then separate again and appear rotated by 90°. As the equilibrium orbits continue to get closer, the internal region of the bucket shrinks continuously and reduces to a point at the limit $\varepsilon = 0$ (fig. 44e). When the critical orbit passes through the equilibrium orbit, the whole picture recommences in the reverse order (fig. 44, f, g, h). During this the point $-|\varphi_s|$ of the equilibrium orbit, which corresponds to the acceleration, becomes the saddle-point, and the point $+\varphi_s$ becomes the centre.

One should remember that the nearer one gets to the critical point, the less accurate the picture becomes, as any finite velocity of variation of the parameter $\mathcal{H}(\tau)$ will destroy the initial assumptions of adiabaticity. However, such a manner of presenting the facts gives a rather clear view of the behaviour of the particles near the critical energy, and also suggests some possible ways of crossing it.

As the adiabaticity is destroyed right from the start for large oscillations, particles which lie on the separatrix (the boundary line of the bucket) begin to fall out of the stable region already before one arrives at the critical point. Particles which have small amplitudes, which satisfy condition (7.15), behave according to the expressions (7.7) and (7.11) obtained above. After the transformation of the phase plane they must, evidently, find themselves on the separatrix corresponding to a deceleration of the particles, if one does not apply some external perturbation. As already mentioned, such a perturbation can be an abrupt change of phase of the accelerating voltage by $2|\varphi_s|$. In fact, with such a procedure, the points representing the particles remain where they were, and the whole array of phase trajectories moves so as to fall on the bucket that corresponds to acceleration.

To come to quantitative estimates, let us observe that the phase jump can be realized by introducing a perturbation $\Delta\omega_0(\tau)$ of the radiofrequency, which acts during a time τ_0. Evidently the condition

$$\int_{\tau_0} \Delta\omega_0(\tau)\mathrm{d}\tau = +2|\varphi_s| \,. \tag{7.20}$$

must be fulfilled. Making use of relation (3.18b), one finds without difficulty that the induced force on the right-hand side of eq. (7.5) has now the form

$$F(\tau) = -\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{\tau}(2|\varphi_s|\,\delta(\tau) - \Delta\omega_0(\tau))\right]. \tag{7.21}$$

In the ideal case, if the phase of the voltage changes suddenly and exactly at $\tau = 0$, then $\Delta\omega_0(\tau) = 2|\varphi_s|\delta(\tau)$ and $F(\tau) \equiv 0$. This shows that the critical energy does not perturb the phase motion* and that the phase transfer, remain in the neighbourhood of the new stable point, according to expressions (7.7) and (7.9).

In practice, however, one cannot avoid some inaccuracy in the moment of the phase transfer, as well as a finite length of the transfer process itself. Both have the consequence that $F(\tau) \neq 0$ in the interval τ_0, and the phase oscillations will experience some swing which can be easily calculated. Knowing the solution of the homogeneous equation (7.5) and the explicit expression for the force $F(\tau)$

$$\frac{\mathrm{d}C_{1,2}}{\mathrm{d}w} \doteq \pm \frac{4\pi}{3^{\frac{1}{2}}\Omega_0^2} w^{-\frac{1}{3}} F(\tau) J_{\mp\frac{1}{3}}(w). \tag{7.22}$$

As examples, we shall examine the following perturbations.

(a) Transfer accomplished instantaneously, but too late by $\Delta\tau$, where $\Delta\tau \ll \Omega_0^{-1}$. Then the quantities C_1 and C_2 increase by

$$\begin{aligned} \Delta C_1 &= \frac{3^{\frac{1}{2}}2^{\frac{1}{3}}\pi}{\Gamma(\frac{1}{3})} |\Omega_0 \Delta\tau|\,|\varphi_s|, \\ \Delta C_2 &= -\frac{3^{\frac{1}{2}}\pi}{2^{\frac{1}{3}}\Gamma(\frac{2}{3})} |\Omega_0 \Delta\tau|^3 |\varphi_s|. \end{aligned} \tag{7.23}$$

Thus, the perturbation of the phase oscillations appears to be small if the timing error in the phase transfer is small in comparison with the characteristic time Ω_0^{-1}.

(b) The frequency perturbation has the form of a rectangular pulse

$$\Delta\omega_0(\tau) = \begin{cases} \dfrac{2|\varphi_s|}{\Delta\tau}, & |\tau| \lesssim \tfrac{1}{2}\Delta\tau. \\ 0 \end{cases} \tag{7.24}$$

We obtain the following expressions in a quite similar way:

$$\begin{aligned} \Delta C_1 &= \frac{3^{\frac{1}{2}}\pi}{2^{\frac{1}{3}}\Gamma(\frac{1}{3})} |\Omega_0 \Delta\tau|\,|\varphi_s|, \\ \Delta C_2 &= -\frac{3^{\frac{1}{2}}2^{\frac{1}{3}}\pi}{2^7\Gamma(\frac{2}{3})} |\Omega_0 \Delta\tau|^3 |\varphi_s|, \end{aligned} \tag{7.25}$$

* This refers, of course, only to particles with small amplitudes for which (7.15) is satisfied. In practice it is just their behaviour which presents the greatest interest.

from which it follows again that the length of the transfer must be much smaller than Ω_0^{-1}. We note that if in both these cases we write $\Delta\omega_0 = 0$, the forced phase oscillations satisfy the condition $\eta_B(0) = -2|\varphi_s|$, as must be the case in the absence of phase jump.

So the tolerances at the moment of the jump and in the course of the whole process do not appear to be especially tight. However, the effect of the various perturbations on the particle motion grows considerable near the critical point. In fact, in general, when the frequency of the accelerating field changes a little the equilibrium energy and the length of the equilibrium orbit also vary by a small amount. But near the critical point the frequency does not in practice depend on energy, and small discrepancies of frequency and field can be compensated only by a large variation of the equilibrium energy. In other words a small inaccuracy $\Delta\omega_0$ leads to a large displacement of the equilibrium orbit. Quite similar is the action near the critical point of field variations which do not follow ω_0: an instantaneous increase of the field in the whole magnet leads to an increase of revolution frequency, which can be brought into agreement with ω_0 only by a large change in the equilibrium energy.

A modification of the equilibrium energy means a displacement of the point of equilibrium phase oscillations and consequently an increase of these. One can easily show that this increase is described, in the frame of eqs. (7.5), by relations of the form (7.22)

$$\frac{dC_{1,2}}{dw} = \pm \frac{2\pi}{3^{\frac{1}{2}}\Omega_0} J_{\pm\frac{1}{3}}(w) w^{\frac{1}{2}} \frac{d}{dw} w^{-\frac{1}{2}} \Delta\omega_0 . \tag{7.26}$$

(We consider only the perturbation $\Delta\omega_0$, but this applies as well to the perturbation $\Delta\langle B_z\rangle_s$, if one exchanges $\Delta\omega_0/\omega_0$ for $-\alpha\Delta\langle B_z\rangle_s/\langle B_z\rangle_s$ in the expressions obtained.) If the perturbation is for instance constant in the critical region we obtain, for the increase of the quantities $C_{1,2}$, the following simple expression

$$\Delta C_{1,2} = \frac{4\pi\Delta\omega_0}{3^{\frac{1}{2}}|\Omega_0|} \int_0^\infty J_{\pm\frac{1}{3}}(w)\,dw = \frac{4\pi\Delta\omega_0}{3^{\frac{1}{2}}|\Omega_0|} . \tag{7.27}$$

Thus, the perturbation of the radiofrequency should be much less than the characteristic frequency Ω_0, that is

$$\frac{\Delta\omega_0}{\omega_0} \ll \left[\frac{eV\cos\varphi_s}{3\pi E_s}\right]^{\frac{1}{2}} \left[\frac{\alpha q}{1-\alpha}|\operatorname{tg}\varphi_s|\right]^{\frac{1}{2}} . \tag{7.28}$$

For large machines this tolerance may appear very severe. For instance, for $E_k = 6.7$ GeV, $\alpha = 2\times10^{-2}$, $eV = 20$ kV, the discrepancy between frequency and field should not exceed 10^{-5} to 10^{-6} (in terms of frequency). The tolerance on the field is larger by an order of magnitude.

Because of these severe tolerances required for the tuning of the radiofrequency to the field in the neighbourhood of transition, use is made of systems where the parameters are regulated by the beam itself [24–26]. There the programme established in advance provides only a relatively coarse regulation of the frequency. The precise tuning is done by means of a special servo-system based on the direct measurement of the beam revolution frequency. As experience shows, such a scheme makes it possible to relax considerably the tolerance on the concordance of frequency and field. Its realization in practice is, however, an intricate task in itself.

To end this section, let us dwell again on the possibility of eliminating the critical energy completely in a proton accelerator by making its magnet more complicated.

To this end, as expression (7.2) has a quite general character, one can reduce α so much that the critical energy will exceed the maximum energy of the accelerated particles. Moreover, if one makes the quantity α negative, the critical energy is removed once and for all. The first work drawing attention to this point was [27], in which a new type of strong-focusing accelerator was proposed, the ring-phasotron (see ch. 7).

Let us come back to expression (1.13). We have already said that if the orbit curvature K and the envelope of betatron oscillations $|f|$ depend very little on azimuth, the quantity α lies near to ν_x^{-2}, is positive and cannot be substantially reduced. For this reason, one can influence α only by distorting the orbit, introducing for instance sectors with a negative curvature. However, any distortion of the orbit with respect to the circular one reflects itself on the dimensions of the accelerator, in such a way that if the maximum magnetic field value is given, the maximum particle energy which corresponds to a given orbit length is decreased. In doing this it is important to modify the orbit in an optimum manner in order to obtain the maximum possible change in α with a relatively unimportant increase in orbit length. As Vladimirsky and Tarasov have shown [28, 29], one can achieve this by introducing a resonance perturbation with a harmonic number $k \approx \nu_x$. Then in the sum (1.13) the term with $k \approx \nu_x$ can be made equal to and even larger than the quantity $-\nu_x^2$, in other words one can make α equal to zero or even negative. One should stress, however, that with such a method the quantity ν_x must be chosen near an integral number, a circumstance which aggravates the effect of magnetic field perturbations and requires that one should pay special attention to this correction. The required modifications to the equilibrium orbit may be accomplished by special magnets with reversed field direction distributed regularly along the circumference. Calculations show that one can eliminate the critical energy only at the price of a noticeable increase of the mean radius of the machine (of the order of 15–20%).

There are still other ways to overcome the critical energy. It is possible in principle, for instance, to switch off the high-frequency voltage near the critical point and to pass the particles through it in a betatron regime

by means of a special core. After the transition, the high-frequency voltage is switched on again and the particles continue to be accelerated in the self-focusing regime. There is quite a large number of possibilities of applying various types of acceleration when one has a magnetic field constant in time (see ch. 7 § 4).

8. THE INDUCTION REGIME OF ACCELERATION

The induction accelerator or betatron has played an important part in the development of accelerator techniques, and it has not lost its importance today. Its basic feature is that the accelerating electric field, which is rotational, is produced by a magnetic flux increasing with time which is contained within the orbit. This flux is usually chosen in such a way that the orbit radius remains approximately constant in the course of acceleration.

The theory of the azimuthally symmetrical betatron (see ch. 1 § 4) is nowadays well established and treated in well-known reports and review papers (see for instance [30–33]). However, the field of application of the betatron regime is much wider and may include accelerators with fixed magnetic guide field and non-circular orbits. In this connection we shall examine some general laws of the betatron regime which do not depend on the practical form of the field or of the orbit geometry.

Usually one may consider that the accelerating field does not change appreciably during the time of one revolution. So

$$\langle \mathscr{E}_\tau(\boldsymbol{r}_0(t, E))\rangle = \frac{1}{\Pi} \oint \mathscr{E}_{\text{ind}} \, \mathrm{d}\sigma \,, \tag{8.1}$$

where $\mathrm{d}\sigma$ is an element of arc, and where the integration takes place along the orbit of length Π. Taking into account that

$$\operatorname{curl} \mathscr{E} = -c^{-1}(\partial B/\partial t),$$

we find by Stokes' theorem

$$\langle \mathscr{E}_\tau \rangle = -\frac{1}{c\Pi} \frac{\partial \Phi}{\partial t} = -\frac{S}{c\Pi} \frac{\overline{\partial B_z}}{\partial t}, \tag{8.2}$$

where Φ is the scalar magnetic flux linking the orbit and w the surface of which the orbit is the boundary. The quantity $\bar{B}_z = \Phi/S$ is thus the

mean magnetic field inside the orbit. Introducing this into (2.7) we obtain a relationship which describes the regime of acceleration in the betatron

$$\frac{dE}{dt} = -\frac{e\omega(E)}{2\pi c} S \frac{\overline{\partial B_z}}{\partial t}. \tag{8.3}$$

As according to (1.11) of ch. 2, $p = -(e\langle B_z\rangle \Pi)/(2\pi c)$, we obtain from (8.3)

$$\frac{d}{dt}[\Pi\langle B_z\rangle] = \frac{2\pi S}{\Pi} \frac{\overline{\partial B_z}}{\partial t}. \tag{8.4}$$

If the conditions are chosen so that the orbit does not vary during acceleration, then Π is constant and

$$\frac{\partial\langle B_z\rangle}{\partial t} = \frac{2\pi S}{\Pi^2} \frac{\overline{\partial B_z}}{\partial t}. \tag{8.5}$$

Let us remember that we have on the left-hand side the field averaged along the orbit, and on the right-hand side the same averaged over the surface enclosed by the orbit. In the case of the rotationally symmetrical field, considered in ch. 1, relation (8.5) takes a particularly simple form (called the 2:1 relation)

$$\frac{\partial B_z}{\partial t} = \frac{1}{2} \frac{\overline{\partial B_z}}{\partial t}. \tag{8.6}$$

The increase of the magnetic field on this orbit thus balances exactly the increase of particle energy. Sometimes, this orbit is called the betatron equilibrium orbit and the corresponding energy the equilibrium energy $E_s(t)$.

It is interesting to look at the orbit behaviour for energies which differ a little from the equilibrium value. For this we rewrite (8.3) in the form

$$\frac{dp}{dt} = -\frac{e}{c\Pi} \frac{\partial \Phi}{\partial t}, \tag{8.7}$$

where $p = Ev/c^2$ is the momentum of the particle, and Φ the scalar flux through its orbit. If we designate the equilibrium quantities by the subscript s and the deviations from the equilibrium values by $\Delta p, \Delta\Pi$

and $\Delta\Phi$ we obtain from (8.7)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta p = -\frac{\Delta\Pi}{\Pi_s}\frac{\mathrm{d}p_s}{\mathrm{d}t} - \frac{e}{c\Pi_s}\frac{\partial\Delta\Phi}{\partial t} = -\alpha\frac{\Delta p}{p_s}\frac{\mathrm{d}p_s}{\mathrm{d}t} - \frac{e}{c\Pi_s}\frac{\partial\Delta\Phi}{\partial t}. \quad (8.8)$$

However,

$$\frac{\partial\Delta\Phi}{\partial t} = \oint \frac{\partial \boldsymbol{B}_s}{\partial t}[\boldsymbol{\rho\tau}]\frac{\mathrm{d}\vartheta}{K_0} = \oint \frac{\partial B_{zs}}{\partial t}\psi(\vartheta)\frac{\Delta p}{p_s}\frac{\mathrm{d}\vartheta}{K_0^2}, \quad (8.9)$$

where the integration is to be performed along the equilibrium orbit. As the form and dimensions of the equilibrium orbit do not change,

$$\frac{\partial B_{zs}}{\partial t} = -\frac{Kc}{e}\frac{\mathrm{d}p_s}{\mathrm{d}t} \quad (8.10)$$

(see ch. 2 § 1). Making use of (8.10) and (8.9) one finds, with (8.8)

$$\frac{\mathrm{d}\Delta p}{\mathrm{d}t} = -\alpha\frac{\mathrm{d}p_s}{\mathrm{d}t}\frac{\Delta p}{p_s} + \frac{1}{\Pi_s}\frac{\mathrm{d}p_s}{\mathrm{d}t}\frac{2\pi}{K_0}\frac{\Delta p}{p_s}\alpha \equiv 0. \quad (8.11)$$

Thus, in a betatron with a fixed equilibrium orbit the deviation of particle momentum from its equilibrium value is an integral of the motion independently of orbit geometry and type of focusing. This means that the momentum spread of the particles in the beam at the end of acceleration remains the same as at injection. As regards the lengths of the orbits of different particles, they satisfy the simple condition

$$\frac{\Delta\Pi}{\Pi_s} = \alpha\frac{\Delta p}{p_s} = \alpha\frac{\Delta E}{\beta_s^2 E_s}. \quad (8.12)$$

In particular, in a betatron with rotational symmetry

$$\frac{\Delta R}{R} = \frac{1}{1-n}\frac{\Delta E}{\beta_s^2 E_s} \approx \frac{1}{1-n}\frac{\Delta E}{E_s}, \quad (8.13)$$

where the approximation is exact for the relativistic case ($\beta_s \approx 1$). This shows that all particle orbits tend, proportionally to $\langle B_z\rangle^{-1}$, towards the equilibrium orbit which is sometimes called the limit orbit for this reason. The mechanism of this 'damping' is closely related to the fact that in a betatron having a limit orbit all particles experience equal energy gains per unit path length. In other words, the manner in

which the guide field varies with time is such that the average electric field is the same for all orbits which are close to the equilibrium one.

In the case of the betatron having a limit orbit, one can replace $\partial\Phi_s/\partial t$ by $\mathrm{d}\Phi_s/\mathrm{d}t$ and integrate eq. (8.7). One then obtains a simple relation between the total increase of the equilibrium momentum in the course of acceleration and the total variation of the flux enclosed by the equilibrium orbit

$$p_{s,\,\mathrm{final}} - p_{s,\,\mathrm{initial}} = \frac{e}{\Pi_s c} \left| \Phi_{s,\,\mathrm{final}} - \Phi_{s,\,\mathrm{initial}} \right| . \tag{8.14}$$

In the betatron with fixed guide field the rotational accelerating field is also produced by a special core, through which one forces a varying magnetic flux $\Phi_0(t)$. Here there is no limit orbit of course, as all particles move to larger radii in the course of acceleration*. The momentum spread is evidently not an integral of motion in such a case. In fact, as all particles are accelerated by the action of the same variable magnetic flux Φ_0, $\partial\Phi_0/\partial t$ does not depend on E, so that from (8.3) follows

$$\frac{1}{\omega(E)} \frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{e}{2\pi c} \frac{\partial\Phi_0}{\partial t} = -\frac{e}{2\pi c} \frac{\mathrm{d}\Phi_0}{\mathrm{d}t} . \tag{8.15}$$

Thus, in a fixed guiding field all the particles receive the same energy gain per turn, so that the integral of motion is now the quantity

$$\Delta u = \text{const.} , \tag{8.16}$$

where

$$u = \int \frac{\mathrm{d}E}{\omega(E)} . \tag{8.17}$$

Relation (8.16) is quite generally characteristic of acceleration in fixed fields (see ch. 7 § 4). If the energy spread is small enough, one may consider that

$$\Delta E \approx \text{const.}\ \omega(E) , \tag{8.18}$$

i.e. that the energy spread varies proportionally with the revolution frequency.

* More exactly, to orbits with a larger $\langle B_z/K \rangle$.

The total energy gain can be obtained by simply integrating (8.15)

$$\int_{E_{\text{initial}}}^{E_{\text{final}}} \frac{dE}{\omega(E)} = \frac{e}{c} |\Phi_{0\,\text{final}} - \Phi_{0\,\text{initial}}| = \frac{e}{c} |\Delta\Phi| \tag{8.19}$$

and depends on the practical form of the function $\omega(E)$.

It should be stressed that the betatron with fixed guide field possesses a feature which makes it different from the ordinary betatron or synchrotron, in which the time available for the capture of the particles into the accelerating regime is a very small fraction of the acceleration cycle (10^{-3} to 10^{-4}). In a betatron with fixed guide field it is possible to realize conditions for which the useful capture time is much larger. For instance, for $\Delta\Phi = \Phi_{\max}$, where $\Phi_0 = \Phi_{\max} \sin 2\pi ft$, the fraction of the acceleration time in which capture can take place is $\varepsilon \approx 0.25$, and in general it is (with a sinusoidally varying flux),

$$\varepsilon = \frac{1}{2\pi} \arccos\left(\frac{\Delta\Phi}{\Phi_{\max}} - 1\right). \tag{8.20}$$

Because of this, the beam intensity in such an accelerator can be made substantially larger than in the usual betatrons or synchrotrons, the other conditions remaining the same.

The motion of electrons having a high energy in an accelerator is accompanied by the emission of a distinctive radiation (see ch. 5 § 1) whose power increases extremely quickly with energy, in fact as $B_z^2 E^2$, and whose spectrum is shifted towards short wave-lengths (visible light, ultraviolet, soft X-rays). Because of the radiation losses the condition 2 : 1, which the magnetic field of the betatron obeys, is no longer sufficient for the existence of an equilibrium circular orbit [34]. The equilibrium orbit of the electrons changes into an uncurling spiral the pitch of which is, according to (8.13),

$$\Delta R = -\frac{R}{1-n} \frac{\Delta W_{\text{turn}}}{E}, \tag{8.21}$$

where ΔW_{turn} are the losses per turn. The permissible limit ΔR for the variation of R is determined by the radial width $\Delta R_{\max}$ of the working region in the magnetic field.

From the condition

$$\int_0^t \frac{R}{1-n} \frac{\Delta W_{\text{turn}} \omega(E)}{2\pi} \, \mathrm{d}t < \Delta R_{\max} \tag{8.22}$$

one can obtain the maximum value of the energy up to which particles can be accelerated in a betatron. This figure lies around 300 MeV, if the constructional parameters are properly chosen.

To prevent a shrinkage of the equilibrium orbit an account of radiation losses, it is evidently necessary to compensate them with the help of the external accelerating field. For a betatron with rotational symmetry one finds, instead of the 2 : 1 condition, the relation

$$\left| \frac{\partial \bar{B}_z}{\partial t} \right| = 2 \left| \frac{\partial B_z}{\partial t} \right| + \frac{\Delta W_{\text{turn}}}{(e/c)\pi R^2} . \tag{8.23}$$

According to this more general condition, the magnetic flux inside the orbit has to increase faster than before, in order to create an accelerating field that is sufficiently large.

In practice, it is not convenient to compensate the radiation losses by means of the rotational induction field. A much wider range of possibilities of increasing the energy of the accelerated electron exists with the synchrotron. If the voltage amplitude V at the accelerating gap is large enough the position of the equilibrium phase φ_s will, because of the self-focusing, move automatically as the losses increase in order to compensate them. At the same time it is not difficult in principle to increase the amplitude V in the course of acceleration according to the increase of the radiation losses. These and other important problems connected with radiation effects will be examined in ch. 5.

Transition regime

In most synchrotrons one accelerates the electrons in the betatron regime at the beginning of the cycle and right up to relativistic speeds with the rotational electric field created by a special core. At some 'transition' energy, usually of the order of a few MeV, one switches on and maintains an accelerating voltage V in the cavity, as the betatron core saturates. The electrons are captured into the synchrotron regime and accelerated in this manner until the end.

Let us consider briefly the process of passing from betatron to synchrotron acceleration.

This process (transition regime) must take place at the moment when the electron revolution frequency in the betatron regime ($\omega \propto v/r$) becomes nearly equal to the constant frequency of the accelerating voltage on the cavity, $\omega \approx \omega_0$. Under these conditions it is easier for the phase of the particle to make the transition from circular motion to motion oscillating around the equilibrium value (using the pendulum terminology).

It will be remembered that when we derived the phase equation (3.23) we never made the assumption that the coefficients appearing there were constant. So to study the 'transition regime' we can use as before eq. (3.23), where we put $V = V(t)$, $V(t_0) = 0$, $V(t_0 + \tau) = V_{app}$, t_0 being the beginning of the transition and τ the time when the voltage is applied.

It is important to note that in practice we do not need to know the solution of the phase equation. One has only to be able to determine the set of initial conditions (φ_i, E_i) at the end of the betatron cycle, which correspond to the capture into the synchrotron regime. The whole set of initial conditions before the transition regime is, on the phase surface (which is a cylinder), a band

$$0 \leqslant \varphi \leqslant 2\pi\,, \qquad E_{min} \leqslant E \leqslant E_{max}\,, \tag{8.24}$$

where E_{min}, E_{max} are determined by the conditions at injection into the betatron regime and by the value of the energy at which transition takes place.

We shall examine the results obtained in the two limiting cases of fast and slow application of V.

(a) Fast application of V. With the condition $\Omega^{-1}(\mathrm{d} \ln V/\mathrm{d}t) \gg 1$ (instantaneous application) the region of permissible initial conditions $(\varphi_i, E_i)_{perm}$ is delimited by the separatrix (4.2), (4.3), which should be laid on to the rectangle (8.24) for computing the capture efficiency. If V_{app} is sufficiently large and if the instant of switching is correctly chosen the capture efficiency is near to one (if one does not take into account the chamber walls and similar boundary conditions). If $\Omega^{-1}(\mathrm{d} \ln V/\mathrm{d}t) \gtrsim 1$ (rapid application), then the set of conditions $(\varphi_i, E_i)_{perm}$ is somewhat modified with respect to the separatrix.

(b) Slow application of V. With the condition $1 \gg \Omega^{-1}(\mathrm{d} \ln V/\mathrm{d}t) > \mathrm{ctg}\, \varphi_s$ (quasi-adiabatic application) which is violated at the beginning of the transition when $V \approx 0$, the region $(\varphi_i, E_i)_{perm}$ is the rectangle

$$(E_i - E_s)_{perm} = \frac{eV}{2\pi}\left[\omega_s(t - t_1)\cos\varphi_s \pm \sqrt{2\sin\varphi_s}\left(\frac{\omega_s}{\Omega}\right)\right], \tag{8.25}$$

$$0 \leqslant \varphi \leqslant 2\pi\,,$$

where t_1 is the instant when the frequencies ω, ω_0 coincide.

If V_{app} is large enough for the surfaces of the rectangles (8.24) and (8.25) to be approximately equal, the capture of the particles will be practically complete. If $\mathrm{ctg} \gg \Omega^{-1}(\mathrm{d} \ln V/\mathrm{d}t)$ (adiabatic application) the analysis shows that the picture is more complicated and $(\varphi_i, E_i)_{perm}$ takes the form of a system of discrete fields (for details, see [11, 12]).

The main conclusion of the theory of the transition regime is that when V_{app} is large enough but still compatible with the boundary conditions (chamber walls), most of the particles are captured from the betatron into the synchrotron regime, and this result does not depend very much on the form of the dependence $V = V(t)$.

CHAPTER 5

RADIATION EFFECTS IN CYCLIC ELECTRON ACCELERATORS

The principles on which weak- and strong-focusing accelerators of protons and heavy ions are based, are generally valid also for electron accelerators. We find, however, in large electron accelerators specific phenomena related to the fact that the electrons are ultrarelativistic during the acceleration cycle.

The most important phenomenon which affects substantially the construction and regime of operation of the electron accelerator is the existence of an intense electromagnetic radiation emitted by relativistic electrons when moving in a magnetic field. The importance of this radiation in the operation of a cyclic electron accelerator (a betatron) was first stressed in refs. [3, 4]. The necessity of compensating the energy losses due to radiation presents severe restrictions on the accelerating system. As several authors have shown [9–14, 25, 26] the radiation exerts its influence on the whole dynamics of the electron motion, that is on betatron and synchrotron oscillations, and can under various conditions lead to their excitation or damping. It is essential to calculate this influence if one wants to design correctly and operate rationally electron synchrotrons for high energies.

1. BASIC PROPERTIES OF THE RADIATION

The various properties of the radiation emitted by a relativistic electron moving in a magnetic field are given, for example, in [1–8]. Of these properties we shall consider here those which have an immediate bearing on the dynamics of electron motion in an accelerator. With increasing electron energy E, an extremely rapid increase takes place in the power of the energy losses due to radiation: $W \propto E^4$ (when the orbit radius is constant).

If, as is generally the case, the orbit curvature $K(\vartheta)$ does not vary too rapidly, one may write the formula for the 'instantaneous' value of the power in the form

$$W = \tfrac{2}{3}e^2 cK^2\gamma^4 , \tag{1.1}$$

where γ is the relativistic factor

$$\gamma = \frac{E}{m_0 c^2} \gg 1 , \qquad E = \frac{|eB_z|}{K} , \tag{1.2}$$

and m_0 the electron rest mass.

At high energies, the greater part of the power of the accelerating system is used in applying the radiation losses. Thus, for an energy of 10^9 eV the losses per turn amount to ≈ 30 keV/turn, and for a ten times higher energy (about 10 GeV) the losses go up to 20–30 MeV turn. In general, a relativistic electron in one turn of a circular orbit loses by radiation an energy equal to

$$\Delta W_{\text{turn}} = 2\pi(cK)^{-1} W \approx 90KE^4 \text{ keV} , \tag{1.3}$$

where K is expressed in m^{-1}, and E in GeV.

Another feature of the radiation considered is its strong directivity: practically all the radiation is concentrated in a narrow cone with an aperture angle $\propto \gamma^{-1}$ radian around the direction of the instantaneous velocity of the electron. As this angle is extremely small (for $E = 5$ GeV, it amounts in all to a value $\lesssim 1'$) one may consider with sufficient accuracy ($\propto \gamma^{-2}$) that the radiation is directed along the tangent to the trajectory. This extremely unsymmetrical angular distribution plays an important role in the processes taking place in the accelerator.

The spectrum of the radiation of an electron travelling on a circular orbit is also rich in higher harmonics of the revolution frequency. In contrast with the non-relativistic case the maximum of the radiation spectrum in the case considered here lies in the neighbourhood of a frequency ω_c, the harmonic number of which is a very large figure:

$$\omega_c = k_c \frac{2\pi c}{\Pi} , \qquad k_c = \tfrac{3}{2}\gamma^3 . \tag{1.4}$$

For high energies ($\gtrsim 1$ GeV) this corresponds to the region of soft X-rays.

In the case of a circular motion ($K = \text{const.}$) one may describe the radiation losses by the following formulae. The expression for the angular distribution of the intensity of the radiation of the kth harmonic has the form

$$\mathrm{d}W_k = \frac{e^2 cK^2 \beta^2 k^2}{2\pi\gamma^2}[J_k^2(k\beta \sin\sigma)\mathrm{ctg}^2\sigma + J_k'^2(k\beta \sin\sigma)\beta^2]\mathrm{d}o\,, \quad (1.5)$$

where σ is the angle between the direction of the radiation and the z-axis (perpendicular to the orbit plane), J_k and J_k' the Bessel function of kth order and its derivative, and $\mathrm{d}o$ the element of solid angle. The radiated power W_k of the kth harmonic is found by integrating expression (1.5) over all angles

$$W_k = e^2 cK^2 \beta k \left[2\beta^2 J_{2k}'(2k\beta) - (1-\beta^2) \int_0^{2k\beta} J_{2k}(x)\mathrm{d}x \right]. \quad (1.6)$$

Expression (1.6) is not convenient for use in practice as the harmonic number k defines not only the argument, but also the order of the Bessel function. Only in the relativistic case $\gamma \gg 1$ may the formula for W_k be represented with sufficient accuracy in the much more practical form

$$W_k = \frac{e^2 cK^2}{\pi\sqrt{3}} \frac{\beta^2}{\gamma^2} \int_{k/k_c}^{\infty} K_{\frac{5}{3}}(x)\mathrm{d}x\,, \quad (1.7)$$

where the number k_c is determined according to (1.5), $K_{\frac{5}{3}}$ is the MacDonald function of order $\frac{5}{3}$ and K is as before the curvature of the orbit. Making use of asymptotic approximations, we get the following dependence of the radiated power of the kth harmonic on the number K in the relativistic case

$$W_k \approx 0.52 e^2 cK^2 k^{\frac{1}{3}} \quad \text{for} \quad 1 \ll k \ll k_c\,, \quad (1.8)$$

$$W_k \approx 0.35 e^2 cK^2 (k/k_c)^{\frac{1}{3}} \mathrm{e}^{-k/k_c} \quad \text{for} \quad k \gg k_c\,. \quad (1.9)$$

The general form of the W_k spectrum is shown schematically in fig. 45. The total power, given by the surface under the curve W_k, is obtained

by summing (1.6) over k:

$$W = \sum_k W_k = \tfrac{2}{3} e^2 c K^2 \frac{\beta^4}{(1-\beta^2)^2}. \tag{1.10}$$

In the relativistic case $\gamma \gg 1$, this gives a formula of the type (1.1) for which the dependence $E \propto \gamma^4$ is characteristic.

As most of the radiation lies in the region of extremely short wavelengths ($k_c \gg 1$), the formulae derived may be applied in practice, not only to circular orbits, but also to arbitrary electron trajectories, taking for K the curvature of the trajectory at the point considered.

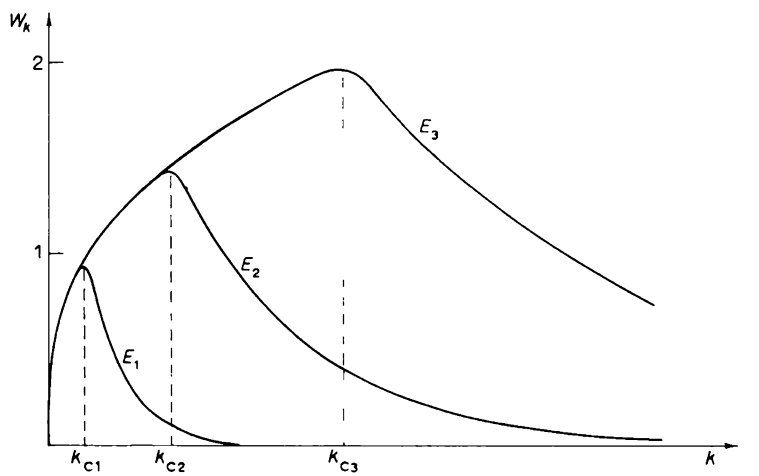

Fig. 45. Radiation spectrum of the relativistic electron for various energies $E_1 < E_2 < E_3$ (schematic).

The properties of the radiation that we have indicated, and in particular formulae (1.1)–(1.10), have been derived on the basis of classical radiation theory and are in good agreement with experiment (see for instance [35]). To have an idea of the limit of validity of these formulae, one may propose the following criterion [7]: the quantum corrections to the main characteristics of the radiation are small if the radiation wave-length is much larger than the de Broglie wave-length of the electron, or, what amounts to the same, if the energy of the electron is much

larger than the energy of the radiation quantum $E \gg \hbar\omega_c$. The reasoning can be made in other ways but they all lead formally to the condition

$$E \ll E_\hbar = E_0 \left(\frac{E_0}{c\hbar K}\right)^{\frac{1}{2}}. \tag{1.11}$$

Strict calculations confirm this criterion. As with the magnetic fields that one usually encounters the energy E will only be comparable with $E_\hbar$ if it reaches values of the order of 10^{15} eV, the quantum corrections to the intensity and spectrum of the emitted radiation are negligible in all practical cases.

In addition, one has in extreme cases also to consider the corrections which must be applied to the classical theory of the electron motion because of the quantum statistical character of the radiation [10–14]. The motion of electrons in a synchrotron can take place only within certain limits given either by the self-focusing regime or by the useful region of the magnetic field. Thus, even relatively small perturbations of the motion, caused by statistical quantum fluctuations of the radiation, may be important. This question will be examined in full in § 5 of this chapter, as well as in § 3 of ch. 6.

Also the fact that in any accelerator there is not just one electron but a whole stream of them, has a bearing on the radiation. One should, to be general, consider the interference of the radiation emitted by the various electrons and the effect of interactions of the latter on the resulting radiation from the beam [5.15]. In the more intense region of the spectrum – the short wave-lengths – the radiation is incoherent, that is the electrons radiate independently and the intensity of the radiation from the beam is proportional to their number S. In the longwave range the radiation of the individual electrons is coherent and in principle the actual intensity can be proportional to S^2.

The characteristic feature of the coherent radiation is that it is independent of the energy of the radiating electron (or of its γ value) on the assumption that $\gamma \gg 1$. This is due to the fact that, as is shown in fig. 45, the longwave part of the spectrum does not vary appreciably when the electron energy changes. Increase of the radiation losses proportionally to γ^4 occurs principally on account of variations of the spectrum which take place in the region of short wavelengths corresponding to $k \gtrsim k_c$. Calculations show that the power of the coherent radiation of a single

electron in a relativistic bunch having an effective angular dimension θ_0 is approximately

$$W_{\text{coh}} \approx e^2 c K^2 \theta_0^{-\frac{4}{3}} S . \tag{1.12}$$

In practice the coherent radiation is strongly damped by the neighbouring conducting surfaces (vacuum chamber walls). Without going into the details of the calculation, let us make some simple qualitative considerations to elucidate this phenomenon [16]. Let us assume that there is a conducting plane parallel to the orbit plane at a distance d from it. We replace this plane by a mirror image of the electron, with the opposite charge. The radiation of the electron is thus equivalent to the radiation of a dipole of moment $2ed$ travelling along a circle with radius R. Evidently the longwave part of the radiation ($\lambda > 2d$) is cancelled out, whereas the shortwave part ($\lambda \ll 2d$) is not changed at all. Calculations show that the coherent radiation of an electron beam decreases in the presence of a conducting plane approximately as $(R/d)^2$, which is extremely fast. For usual beam parameters, the coherent radiation losses are on account of the screening negligibly small in comparison to the losses due to the incoherent radiation.

So we shall consider in future only the questions related to the effect of radiation on the motion of a single electron without considering interference corrections. It is only at very high intensities ($S = 10^{13}$ to 10^{14}) which have not yet been reached, and under particular conditions that the effect of the coherence can begin to be felt (for more details see [17–19]).

2. EQUATION OF MOTION OF AN ELECTRON IN PERIODIC MAGNETIC FIELDS IN THE PRESENCE OF RADIATION

Let us consider an arbitrary periodic magnetic system with a synchrotron regime of acceleration in which the orbit curvature $K(\vartheta)$ and the magnetic field index $n(\vartheta)$ are periodic functions of the generalized azimuth ϑ. The equations of the electron motion in the neighbourhood of the equilibrium orbit without emission of radiation were derived in the earlier chapters (see ch. 2, eq. 2.13 and ch. 4, eq. 3.22). In order to take radiation effects into account, one has to supplement these equa-

tions with the terms describing the interaction force caused by the emission of radiation.

To begin with we take the four-dimensional classical equation of the motion of an electron experiencing the reaction of radiation emission

$$m_0 c \frac{\mathrm{d}u_i}{\mathrm{d}s} = \frac{e}{c} F_{ik} u_k + \frac{2e^2}{3c}\left(\frac{\mathrm{d}^2 u_i}{\mathrm{d}s^2} + u_i u_k \frac{\mathrm{d}^2 u_k}{\mathrm{d}s^2}\right), \tag{2.1}$$

s being the path, u_i the components of the velocity and F_{ik} the components of the electromagnetic field tensor. The radiation reaction force is given by the space components of the two terms in brackets. As is shown, for instance, in [6], the second term $(2e^2/3c)u_i u_k(\mathrm{d}^2 u_k/\mathrm{d}s^2)$ happens to be in the relativistic case (in orders of magnitude) γ^2 times larger than the first term $(2e^2/3c)(\mathrm{d}^2 u_i/\mathrm{d}s^2)$, which describes the weak damping caused by the force proportional to $\ddot{x}$. We notice that at the non-relativistic limit for $v \to 0$, the picture changes: the second term, in the absence of the first one, becomes vanishingly small. As only the relativistic case $\gamma \gg 1$ interests us, we need only investigate the second term which is proportional to the square of the acceleration and has in three-dimensional terminology the form

$$\boldsymbol{F}_{\mathrm{rad}} \approx -\frac{2e^4\gamma^2 \boldsymbol{v}}{3m_0^2 c^5}\left\{\left(\boldsymbol{\mathscr{E}} + \frac{1}{c}[\boldsymbol{v}\boldsymbol{B}]\right)^2 - \frac{1}{c^2}(\boldsymbol{\mathscr{E}}\boldsymbol{v})^2\right\}, \tag{2.2}$$

$\boldsymbol{v}$ being the electron velocity and $\boldsymbol{\mathscr{E}}$, $\boldsymbol{B}$ the electric and magnetic fields. Evidently this expression may be rewritten in the form

$$\boldsymbol{F}_{\mathrm{rad}} \approx -\frac{W}{c^2}\boldsymbol{v}, \tag{2.3}$$

where W is the instantaneous power of the radiation losses at the point where the electron is. As the electric field is always small in comparison with the magnetic field in an accelerator, W is expressed sufficiently accurately by formula (1.1) in the case considered.

We have seen above that in the relativistic case one can consider with a precision up to terms of the order γ^{-2} that the direction of the radiation coincides with the direction of the instantaneous electron velocity. So the physical sense of formulae (2.2) and (2.3) is quite clear as they

show that the interaction force appearing as a reaction of the radiation emission is directed in the relativistic case in a direction opposite to the electron motion.

On the basis of what we have said, one may write the equation of relativistic electron motion considering radiation (2.1) in a form which is a generalization of eq. (2.1) of ch. 4 we met not long ago

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = \frac{e}{c}[\boldsymbol{v}\boldsymbol{B}] + e\boldsymbol{\mathscr{E}} - \frac{W}{c^2}\boldsymbol{v}\,. \tag{2.4}$$

We shall use the natural system of co-ordinates (x, z, ϑ) associated with the fixed equilibrium orbit. As was shown in ch. 4 §§ 3 and 5, the action of the transverse components $\mathscr{E}_x, \mathscr{E}_z$ cannot lead to a systematic modification (damping) of the oscillation amplitude. As regards the high-frequency accelerating field $\boldsymbol{\mathscr{E}}$, we shall assume for this reason that there exists only an 'azimuthal' component of this field in the direction along the equilibrium orbit

$$\mathscr{E}_\vartheta \neq 0\,, \qquad \mathscr{E}_{x,z} \equiv 0\,, \tag{2.5}$$

where we can express $\mathscr{E}_\vartheta$ as usual as a travelling wave with a phase φ

$$\mathscr{E}_\vartheta = \frac{V\cos\varphi}{\Pi}\,; \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = q(\omega_\mathrm{s} - \omega)\,. \tag{2.6}$$

Let us consider the motion near the equilibrium orbit lying in the symmetry plane $z = 0$ of the magnetic field, and let us limit ourselves to terms linear in the small quantities $Kx, Kz, \Delta E/E_\mathrm{s} = (E - E_\mathrm{s})/E_\mathrm{s}$. The index s, as in the preceding paragraphs, refers to the stationary values. Let us calculate, according to (1.1), the dependence of the radiation loss power W on the variables x and ΔE

$$W(x, \Delta E, \vartheta) = W_\mathrm{s}(\vartheta)\left[1 - 2n(\vartheta)K(\vartheta)x + 2\frac{\Delta E}{E_\mathrm{s}} + \ldots\right], \tag{2.7}$$

where, starting from the definition of the field index, we have used the relation $B = B_\mathrm{s}(1 - nKx)$. In future we shall drop the index s where this cannot lead to misunderstandings.

With (2.5) we obtain from (2.4) the linearized equations for the x- and z-motion in the form

$$x'' + (\Gamma_1 + \Gamma)x' + g_x x = \frac{K}{K_0^2}\frac{\Delta E}{E_s}, \tag{2.8}$$

$$z'' + (\Gamma_1 + \Gamma)z' + g_z z = 0, \tag{2.9}$$

where g_x, g_z have the usual meaning (see ch. 2, eq. 2.13) and the dimensionless coefficients Γ, Γ_1 are respectively equal to

$$\Gamma = \frac{W}{\omega E} = \frac{2}{3}\frac{e^2}{m_0 c^2}\frac{K^2(\vartheta)}{K_0}\gamma^3, \tag{2.10}$$

$$\Gamma_1 = \frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}\vartheta}. \tag{2.11}$$

The quantity Γ_1 gives the usual adiabatic variation (increase) of the energy in the synchrotron*, and Γ characterizes the reaction force due to the radiation (emission momentum). When the motion occurs in a periodic magnetic system the slow variation of $\Gamma(\vartheta)$, which takes place as the energy E increases, is accompanied by fast oscillations having a period equal to that of the orbit, because of the dependence $K^2(\vartheta)$.

It will be noticed that in eqs. (2.8) and (2.9) we have dropped in the coefficients before x and z small terms containing Γ^2 and Γ' which bring about some change in the frequencies ν_x and ν_z of the oscillations and in their form (that is the Floquet functions). This change is, as a rule, not important, but if necessary the computation can easily be made with the help of the formulae obtained in chapters 2 and 3.

To make clear the physical sense of the different terms in eqs. (2.8) and (2.9) let us stress, first, that according to (2.2) to (2.4) the reaction force of the radiation has components directed against all existing momenta: azimuthal, radial and vertical, whereas the accelerating system tends to maintain (more exactly, to increase) the azimuthal momentum only. As a result, there appears a vertical force proportional to

* In general $\Gamma_1 \propto (\mathrm{d}/\mathrm{d}\vartheta) \ln \langle B_z \rangle$.

$-\Gamma z'$ playing the role of a friction, which we shall for brevity call radiative friction. In the equation of the radial motion (2.8), the origin of the term $\Gamma x'$ is the same as that of the term $\Gamma z'$ in the equation of the vertical oscillations (2.9). The basic difference of eq. (2.8) is that it is inhomogeneous and that the term ΔE appears on the right-hand side. The variation in ΔE due to radiation leads to the fact that the radial betatron oscillations are not damped in the same way as the vertical ones.

We write the third equation of motion, corresponding to the azimuthal direction, in a form which characterizes the rate of change of the electron energy $E \simeq pc$

$$\frac{\mathrm{d}E}{\mathrm{d}\vartheta} = \left(\frac{\mathrm{d}E}{\mathrm{d}\vartheta}\right)_{\mathrm{acc}} + \left(\frac{\mathrm{d}E}{\mathrm{d}\vartheta}\right)_{\mathrm{rad}} . \tag{2.12}$$

The first term on the right describes the action of the accelerating electric field, whereas the second term is related to the presence of radiative losses. As in ch. 4, one can replace the quantity $(\mathrm{d}E/\mathrm{d}\vartheta)_{\mathrm{acc}}$ by its mean value per turn (see 2.6):

$$\left(\frac{\mathrm{d}E}{\mathrm{d}\vartheta}\right)_{\mathrm{acc}} = \frac{eV\cos\varphi}{2\pi} + \ldots \tag{2.13}$$

with an accuracy up to the corrections which take care of the coupling of betatron and synchrotron oscillations, and the discreteness of the accelerating gaps. The conclusion at which we arrived in ch. 4, §§ 3 and 5, that these corrections are of no effect in practice, remains true with radiation emission the effect of which is given in eq. (2.12) by the term $(\mathrm{d}E/\mathrm{d}\vartheta)_{\mathrm{rad}}$. According to the formulae of ch. 2 § 2, defining the natural system of co-ordinates, we have

$$\left(\frac{\mathrm{d}E}{\mathrm{d}\vartheta}\right)_{\mathrm{rad}} = \frac{1}{\vartheta}\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = -\frac{W(x, \Delta E, \vartheta)}{\omega_s}(1 + Kx) , \tag{2.14}$$

where $W(x, \Delta E, \vartheta)$ is the instantaneous power loss of the electron having an energy $E_s + \Delta E$ in a given point x.

Starting from (2.7) and (2.10) to (2.14), the third equation of motion can be written in the form

$$\frac{1}{E_s}\frac{dE}{d\vartheta} = \frac{eV\cos\varphi}{2\pi E_s} - \Gamma_s(\vartheta)\left[1+(1-2n)Kx+2\frac{\Delta E}{E_s}\right]. \quad (2.15)$$

Let us substract from (2.13) the same equation written for a particle travelling along the equilibrium orbit without betatron oscillations and linearize in terms of the variable

$$\eta = \varphi - \varphi_s . \quad (2.16)$$

As a result we get the equation

$$\frac{1}{E_s}\frac{d\Delta E}{d\vartheta} = -\frac{eV\sin\varphi_s}{2\pi E_s}\eta - \Gamma_s(\vartheta)\left[(1-2n)Kx+2\frac{\Delta E}{E_s}\right]. \quad (2.17)$$

Let us note that the following equation is satisfied for the equilibrium particle

$$\frac{eV\omega_s\cos\varphi_s}{2\pi} = \langle \dot{E}_s + W_s \rangle , \quad (2.18)$$

expressing the fact that the whole power supplied by the accelerating system is used to increase the electron energy as the magnetic field grows and to compensate the mean radiation losses. In large electron accelerators the second term is much larger than the first.

To (2.8) and (2.17) one should add the following relation obtained by linearization of (2.6):

$$\frac{d\eta}{d\vartheta} = qK(\vartheta)x . \quad (2.19)$$

It relates together the quantities η' and x. In the next paragraph, starting from eqs. (2.8), (2.9), (2.17) and (2.19), we shall study the effect of the radiation losses compensated by the electric field on synchrotron and betatron oscillations.

3. DECREMENTS OF BETATRON AND SYNCHROTRON OSCILLATIONS

As will be shown, the effect of the radiation losses which are compensated by the accelerating electric field, is to decrease or increase gradually the amplitude of betatron and synchrotron oscillations. This change of

amplitude in the course of acceleration can be characterized by decrements $\zeta_s(\vartheta), \zeta_x(\vartheta), \zeta_z(\vartheta)$ relative to synchrotron and betatron (radial and vertical) oscillations, respectively. The decrement $\zeta_j(\vartheta), j = s, x, z$ describes the variation of the amplitude $A_j(\vartheta)$ according to the formula

$$A(\vartheta) = A(0) \exp\left(-\int_0^{\vartheta} \zeta(\xi) \mathrm{d}\xi\right), \tag{3.1}$$

or

$$A(\vartheta) = A(0) \exp\left(-\int_0^{\vartheta} \langle\zeta\rangle \mathrm{d}\xi\right), \tag{3.1a}$$

where the sign $\langle\ \rangle$ indicates averaging over a period of the system along the azimuth. The quantity $\langle\zeta\rangle$ includes a slow dependence on ϑ (or t) due to the variation of energy during the acceleration process.

The radiation losses of the electron in the accelerator over one turn or one period of synchrotron oscillation is always a very small fraction of the full energy $\Delta W_{\text{turn}}/E = 2\pi\Gamma \ll 1$. Correspondingly, the calculation of the decrements will be performed with the assumption

$$\Gamma \ll \nu_s \ll \nu_{x,z}\,, \tag{3.2}$$

which states that the damping coefficients are small in comparison with the betatron and synchrotron oscillation frequencies ν_x, ν_z, ν_s.

It must be stressed, however, that in spite of (3.2) the amplitude damping described by formulae of the type (3.1) and (3.1a) may in large electron accelerators play an extremely important role. This is due to the fact that in practice the characteristic damping time $\tau \propto 1/\langle\zeta\rangle\omega_s$ may be considerable smaller than the acceleration time T_{acc}, so that oscillation amplitudes may change substantially in the presence of radiation during the acceleration process.

This may be seen quite clearly in the example of vertical oscillations, with which we shall begin our investigation. As follows immediately from (2.9), the decrement of the z-oscillation is simply

$$\langle\zeta_z\rangle = \tfrac{1}{2}\langle\Gamma\rangle\,, \tag{3.3}$$

where Γ is defined by formula (2.10). Consequently, when there are radiation losses and compensation of them the z-oscillations experience

positive damping $A_z \propto \exp(-\frac{1}{2}\int \langle \Gamma \rangle \mathrm{d}\vartheta)$ which is much more important at high energies than the usual damping, which depends on the increase of the field (or energy), and is of the type $A_z \propto B^{-\frac{1}{2}}$ (or $\propto E^{-\frac{1}{2}}$).

Let us consider, to make an estimate, that the energy E grows linearly with time. We then obtain from (1.1) the following law for this effect, which we shall for brevity call 'radiative friction':

$$A_z \propto \exp\left(-\frac{T_{\mathrm{acc}}}{\tau_z}\right), \qquad \tau_z \approx \frac{1.6 \times 10^4}{B_{\max}^2 \gamma_{\max}}, \tag{3.4}$$

where T_{acc} is the acceleration time of the electrons up to the top energy corresponding to the values $\gamma_{\max}, B_{\max}$ (in kilogauss) and τ_z is the characteristic damping time. For instance, for $\gamma = 8 \times 10^3$ ($E_{\max} \approx 4$ GeV) one gets $\tau_z = 0.02$ s, so that with an already small acceleration time T_{acc} of 0.1 s the damping of the amplitude amounts to a factor of about 100. 100. With an acceleration time T_{acc} of 0.2 s the damping factor would already reach about 10^4 in the absence of the unavoidable perturbations.

Let us proceed to the calculation of the decrements ζ_x and ζ_s of radial betatrons and of synchrotron oscillations which is a more complicated task than finding ζ_z. As we see from (2.8) and (2.17), the radiation introduces an additional coupling between betatron and synchrotron oscillations, so that ζ_x and ζ_s have to be determined from the solutions of eqs. (2.8), (2.17) and (2.19) (see also appendix D).

Let us first determine the decrement of betatron oscillations. To exclude the effect of synchrotron oscillations let us consider the motion of the equilibrium particle. Under these conditions an energy deviation $\Delta E/E$, as follows from (2.17) and (2.19), introduces small ($\propto \Gamma$) and fast oscillations with a frequency ν_x. Thus eq. (2.8), roughly speaking, describes betatron oscillations around the equilibrium position, which itself is oscillating with exactly the same frequency and extremely small amplitude. So depending upon the phase situation the betatron oscillations can either increase or be damped. One should also consider the dissipative term in eq. (2.8) which always leads to a decrease of amplitude.

In connection with what we have said above, let us express the quantity x in the form

$$x = y + \frac{\psi(\vartheta)}{K_0} \frac{\Delta E}{E_s}, \tag{3.5}$$

where the function $\psi(\vartheta)$ obeys eq. (1.9) of ch. 4. Introducing (3.5) in (2.8) and (2.7), considering that $\Delta E/E$ has the order of Γ and neglecting terms of order Γ^2, we obtain

$$y''+(\Gamma_1+\Gamma)y'+g_x y=-\frac{2\psi'}{K_0}\frac{\Delta E'}{E_s}-\frac{\psi}{K_0}\frac{\Delta E''}{E_s}, \tag{3.6}$$

$$\frac{\Delta E'}{E_s}=-\frac{eV\sin\varphi_s}{2\pi E_s}\eta-\Gamma(1-2n)Ky. \tag{3.7}$$

It is also easy to convince oneself by a brief calculation that (2.19) is equivalent, for fast phase oscillations and with an accuracy to terms $\propto\Gamma$, to the equation

$$\eta=qK_0[\psi' y-\psi y']. \tag{3.8}$$

Eliminating from the system (3.6) to (3.8) the quantities ΔE and η we arrive at the equation

$$y''+y'\left\{\Gamma_1+\Gamma\left[1-(1-2n)\frac{K\psi}{K_0}\right]+\frac{eVq\sin\varphi_s}{\pi E_s}\psi\psi'\right\}+y[g_x+\mathrm{O}(\Gamma)]=0. \tag{3.9}$$

The component $\mathrm{O}(\Gamma)$ in the last term has an effect only on the form and frequency of the betatron oscillations and leads to a practically negligible shift of the working point. Considering that the average value $\langle\psi\psi'\rangle=0$, we obtain, because of the periodicity of function $\psi(\vartheta)$, from eq. (3.9) the following expression for the decrement of the radiation damping of radial betatron oscillations

$$\langle\zeta_x\rangle=\tfrac{1}{2}\left\langle\Gamma\left[1-(1-2n)\frac{K\psi}{K_0}\right]\right\rangle. \tag{3.10}$$

As we see, because of the action of the radiation and its compensation, radial betatron oscillations experience also the effect of radiative friction. The essential difference in the case of the z-oscillation seen above is the dependence of the sign of this friction on the properties of the magnetic system, that is on the functions $K(\vartheta)$ and $N(\vartheta)$.

To find the damping of synchrotron oscillations let us assume that

the particle travels without betatron oscillation along the orbit corresponding to its energy $E = E_s + \Delta E$. From physical considerations it is clear that for synchrotron oscillations to be damped it is necessary that the transfer of the electron to a new orbit, corresponding to the increase ΔE, should be accompanied by an increase in the power of the radiation losses. This leads to a decrease of ΔE and opposes a further displacement of the electron away from the equilibrium conditions of motion.

In order to obtain the phase equation in the presence of radiation, let us take the average of expression (2.15), taking into account the fact that the motion occurs along the orbit, so that

$$x = \frac{\psi}{K_0} \frac{\Delta E}{E_s}. \tag{3.11}$$

Subtracting from (2.15) the same equation written for the equilibrium particle, we get

$$\frac{1}{E_s} \frac{\mathrm{d}\,\Delta E}{\mathrm{d}\vartheta} = \frac{eV}{2\pi E_s} (\cos\varphi - \cos\varphi_s) - \frac{\Delta E}{E_s} \left\langle \Gamma \left[2 + (1-2n) \frac{K\psi}{K_0} \right] \right\rangle, \tag{3.12}$$

or, eliminating ΔE, with the help of (2.19) and (3.11)

$$\frac{1}{E_s} [E_s \varphi']' + \left\langle \Gamma \left[2 + (1-2n) \frac{K\psi}{K_0} \right] \right\rangle \varphi' = \frac{q\alpha eV}{2\pi E_s} (\cos\varphi - \cos\varphi_s) \tag{3.13}$$

Thus the decrement of synchrotron oscillations is equal to*

$$\langle \zeta_s \rangle = \tfrac{1}{2} \left\langle \Gamma \left[2 + (1-2n) \frac{K\psi}{K_0} \right] \right\rangle \tag{3.14}$$

and depends also on the shape of the magnetic field.

We have given here a particular derivation of the expressions for $\langle \zeta_x \rangle$ and $\langle \zeta_z \rangle$ because it is physically easy to follow. A more elaborate

* It is not difficult to show that for the induction accelerator (betatron) energy deviations ΔE from the equilibrium value are damped with a decrement $\langle 2\zeta_s \rangle$ where ζ_s is given by formula (3.14). Physically this is due to the fact that in the betatron, where there is indifferent equilibrium instead of an azimuthal degree of freedom, the synchrotron oscillations do not exist. The expressions for the decrements $\langle \zeta_x \rangle$ and $\langle \zeta_z \rangle$ remain the same, evidently, as for the synchrotron.

derivation on the basis of the matrix formalism, which is useful from the point of view of a further generalization, is given in appendix D.

Let us see now how radiative friction takes place in some typical magnetic periodic systems.

We treat first the case where the magnetic guide field on the equilibrium orbit does not depend on azimuth, so that the equilibrium orbit curvature K is constant:

$$K = K_0 = \text{const.} \tag{3.15}$$

The instantaneous power of the radiation losses W does not then depend on azimuth either according to (1.1). The expressions for ζ_x, ζ_s become simpler and may be written, when one remembers that in this case $\langle\psi\rangle = \alpha, \langle\psi''\rangle = 0$, in the form

$$\text{(a)} \quad \langle\zeta_x\rangle = -\tfrac{1}{2}\Gamma(1-\alpha), \qquad \text{(b)} \quad \langle\zeta_s\rangle = \tfrac{1}{2}\Gamma(4-\alpha), \tag{3.16}$$

where α is, as before, the coefficient of expansion of the orbit.

In an accelerator with symmetry of revolution, where not only $K = \text{const.}$, but also

$$n = \text{const.}, \qquad \psi = \alpha = \frac{1}{1-n}, \tag{3.17}$$

formulae (3.16) give

$$\text{(a)} \quad \zeta_x = \langle\zeta_x\rangle = \frac{n\Gamma}{2(1-n)}, \qquad \text{(b)} \quad \zeta_s = \langle\zeta_s\rangle = \frac{(3-4n)\Gamma}{2(1-n)}. \tag{3.18}$$

The conditions to have damping of radial betatron oscillations in the case with symmetry of revolution can be expressed, according to (3.18), by the inequalities

$$0 < n < 1, \tag{3.19}$$

which coincide with the stability conditions of radial and vertical (x and z) betatron oscillations in a weak-focusing system.

The damping of synchrotron oscillations takes place in two intervals

$$n < \tfrac{3}{4}, \qquad n > 1, \tag{3.20}$$

between which there is a small region $\frac{3}{4} < n < 1$ where the oscillation is intensified. This region falls inside the interval of stability (3.19).

In practical cases, when the focusing is weak, the decrements $\zeta_z, \zeta_x, \zeta_s$ are not very different from each other as regards their magnitude. For the typical value $n = \frac{2}{3}$ for instance, we have $\zeta_z : \zeta_x : \zeta_s = 1 : 2 : 1$.

If the weak-focusing magnet is cut, i.e. divided in N sectors with an azimuthally symmetric field and an orbit curvature K, and N field-free sections of length l each, the average value of all decrements $\langle\zeta\rangle$ is obtained from (3.18) by replacing Γ by $\langle\Gamma\rangle$.

Let us treat now the very important case of the periodic magnetic system of the strong-focusing synchrophasotron, where $n = n(\vartheta)$, $n \gg 1$. The coefficient of orbit expansion $\alpha \propto |n|^{-1} \ll 1$. As can be seen from (3.16b), under these conditions, synchrotron oscillations in a system with $K = \text{const.}$ will experience an effective damping with a decrement

$$\langle\zeta_s\rangle \approx 2\Gamma . \tag{3.21}$$

Quite different is the picture for the variation of the amplitude of radial betatron oscillations which, according to (3.16a) for $\alpha \ll 1$, have a decrement

$$\langle\zeta_x\rangle \approx -\tfrac{1}{2}\Gamma < 0 , \tag{3.22}$$

with a negative value (in this case it would be more logical to call $|\zeta_s|$ the increment). We can draw the important conclusion first mentioned in [13], that in a strong-focusing synchrophasotron of the usual type the emission of radiation and its compensation do not lead to a decrease of the radial betatron oscillations as for $0 < n < 1$, but to a strong radiative excitation of these with an increment about equal in absolute value.

In connection with this result, let us examine briefly the strong-focusing accelerator of the ring-phasotron type which has the property of scaling orbits (the frequencies ν_x and ν_z are independent of energy). The magnetic field is of the type $B \propto r^{-n_0} F[\theta - h \ln (r/r_i)]$ where n_0, h, r_i are constants and $F(\vartheta)$ is an arbitrary function. This case will be studied in detail in ch. 7 § 3. Here we shall only give the expressions found for the decrements (see [21]).

$$\langle\zeta_x\rangle = \frac{n_0\langle\Gamma\rangle}{2(1-n_0)}, \qquad \langle\zeta_s\rangle = \frac{(3-4n_0)\langle\Gamma\rangle}{2(1-n_0)} . \tag{3.23}$$

These expressions are analogous in their form to those obtained above,

(3.18), for the azimuthally symmetric field if one writes $n = n_0$ therein. However, because of strong focusing $|n_0| \gg 1$ and the resulting effect is similar to the one we found for the strong-focusing synchrophasotron, that is synchrotron oscillations are damped with a decrement $\langle \zeta_s \rangle \approx 2\langle \Gamma \rangle$ and radial betatron oscillations are excited with an increment $|\langle \zeta_x \rangle| = \frac{1}{2}\langle \Gamma \rangle$.

We notice that the expressions (3.3), (3.10) and (3.14) for the decrements $\langle \zeta_z \rangle, \langle \zeta_x \rangle, \langle \zeta_s \rangle$ may be written with the help of the formulae (2.10) for $\Gamma \propto K^2$ in the form

$$\langle \zeta_j \rangle = \tfrac{1}{2}\Gamma_0(\gamma) G_j \qquad (j = x, z, s), \tag{3.24}$$

where $\Gamma_0(\gamma)$ is a general function of the energy (or of γ) applicable in the three cases

$$\Gamma_0 = \tfrac{2}{3} r_e K_0 \gamma^3, \qquad r_e = \frac{e^2}{m_0 c^2}, \tag{3.25}$$

and where the number $G_{x,z,s}$ depends only on the structure of the magnetic periodic system

$$G_z = \left\langle \frac{K^2(\vartheta)}{K_0^2} \right\rangle, \tag{3.26}$$

$$G_x = \left\langle \frac{K^2(\vartheta)}{K_0^2} \left[1 - \frac{K}{K_0}(1-2n)\psi \right] \right\rangle, \tag{3.27}$$

$$G_s = \left\langle \frac{K^2(\vartheta)}{K_0^2} \left[2 + \frac{K}{K_0}(1-2n)\psi \right] \right\rangle. \tag{3.28}$$

Before considering if it is possible to overcome the radiation anti-damping, which will be discussed in the next section, let us examine from a more general point of view the damping (or anti-damping) in the various degrees of freedom of the oscillations related to the radiation and its compensation. We shall state an important feature of the formulae obtained above which refers to the sum of the decrements of the synchrotron and radial betatron motion. This sum is, for any periodic system, independent of the properties of this system, except the mean square of the orbit curvature, and is equal to

$$\langle \zeta_x \rangle + \langle \zeta_s \rangle = \tfrac{3}{2}\langle \Gamma \rangle \tag{3.29}$$

or, which is the same

$$G_x + G_s = 3 \left\langle \left(\frac{K}{K_0} \right)^2 \right\rangle . \tag{3.30}$$

These equations follow immediately from (3.27) and (3.28). If we include in this consideration the vertical oscillation as well, the sum of the decrements of the synchrotron and both betatron oscillations does not depend either on the properties of the system and is

$$\langle \zeta_x \rangle + \langle \zeta_s \rangle + \langle \zeta_z \rangle = 2 \langle \Gamma \rangle . \tag{3.31}$$

This is valid both in the absence of a coupling between vertical and radial betatron oscillations and in the presence of such a coupling (see next section). This feature, expressed in formulae (3.29) to (3.31), is not fortuitous and appears to be a consequence of Jacobi's formulae (see for instance [22]). According to these the Wronskian of the solutions of a system of linear differential equations varies according to the law

$$D(\vartheta) \propto \exp \left[- \int \mathrm{Sp}\, \mathbf{P}\, \mathrm{d}\vartheta \right] , \tag{3.32}$$

where Sp $\mathbf{P}$ is the trace of the coefficient matrix of this system.

Starting from the equations in the form (2.8), (2.9), (2.17) and (2.19) one may, with Jacobi's formulae, obtain for D the following dependence on ϑ for two and three degrees of freedom, respectively

$$D_2 \propto \exp(-3 \int \langle \Gamma \rangle \mathrm{d}\vartheta) \; ; \qquad D_3 \propto \exp(-4 \int \langle \Gamma \rangle \mathrm{d}\vartheta) . \tag{3.33}$$

On the other hand, it follows immediately from the definition of the Wronskian that its variation is given by twice the sum of the decrements for all degrees of freedom which leads to (3.29–3.31). The consequence of this is that any system which increases the damping in one degree of freedom automatically reduces it in the others. As the Wronskian of the solutions decreases, according to (3.33), which means that the four- or six-dimensional phase space occupied by the oscillations shrinks, it is possible, by a suitable damping method, to obtain damping in all three degrees of freedom. Besides this general result, the use of relation (3.29) facilitates calculation of the attenuation decrements of the oscillations when one modifies the periodic magnetic system.

4. METHODS OF DAMPING THE OSCILLATIONS

It follows from (3.10) and (3.14) that magnetic systems which can attenuate radial betatron oscillations without exciting synchrotron oscillations have to satisfy the conditions

$$-\frac{1}{2}\left\langle\frac{K^2}{K_0^2}\right\rangle<\left\langle\frac{K^3(\vartheta)}{K_0^3}\psi(\vartheta)\left[n(\vartheta)-\tfrac{1}{2}\right]\right\rangle<\left\langle\frac{K^2}{K_0^2}\right\rangle, \tag{4.1}$$

where the inequality on the left is the condition for attenuating betatron oscillations, and the one on the right for attenuating synchrotron oscillations. It has also been assumed that $\Gamma\sim K^2$. We have shown above that in strong-focusing systems with $K=\text{const.}$, $|n|\gg 1$ the right-hand inequality is fulfilled, but not the left-hand one.

The purely classical effect considered above of excitation of betatron oscillations by the radiation [13, 14] may, if one does not take special precautions, perturb the normal acceleration regime of the electrons in strong-focusing devices like synchrotrons designed for large energies (1 GeV and above). There is an obvious way of combating the increase of oscillation amplitude by shortening the acceleration time T_{acc}. However, this procedure may be unrealistic for technical reasons.

This is why one should consider other less trivial methods, which, by suitable modifications of the magnetic system, would make it possible to eliminate the radiation antidamping and to attenuate the oscillations. As the qualities and defects of the various modifications are not yet all clear and proved in practice, we shall limit ourselves to a brief and qualitative survey of the question.

Among the proposed methods of damping [14, 24, 25 27], one can distinguish methods based on the redistribution of the decrements of the synchrotron and radial betatron oscillations, and methods making use of the coupling of radial and vertical betatron oscillations. Of course, only such modifications of the magnetic systems can be made as do not alter appreciably the properties of strong focusing, in particular the quantities ν_x, ν_z and the coefficient α.

Let us consider the first group of methods to begin with. As, according to (3.29) the sum of $\langle\zeta_x\rangle$ and $\langle\zeta_s\rangle$ remains constant, it is clear that one can diminish or eliminate the excitation of radial betatron oscillations by reducing the damping of the synchrotron oscillations, that is by making the rate of rise of the power loss W with ΔE smaller. According to formula (1.1), for this one has to change the dependence of the quantity $\langle B^2\rangle\propto\langle K^2\rangle$ upon ΔE, i.e. upon the orbit displacement. This is obtained by a corresponding azimuthal variation of the field which can be realized in different ways.

(1) The first way consists in decreasing the magnetic field B in the radially focusing magnets (in which $n\ll -1$). Let us estimate the value of $\Delta B\ B$ by which one should decrease the field in order to get damping, assuming that the betatron frequencies $\nu_x\approx\nu_z$ are about the same and do not vary when the damping takes place. Computations show that $\langle\zeta_x\rangle$ tends towards zero, that is excitation of oscillations stops, when

$$\left(\frac{\Delta B}{B}\right)_{\mu=\frac{1}{4}\pi}\approx 0.1\,,\qquad \left(\frac{\Delta B}{B}\right)_{\mu=\frac{1}{4}\pi}\approx 0.2\,,\qquad \left(\frac{\Delta B}{B}\right)_{\mu=\frac{1}{4}\pi}\approx 0.25\,.$$

At the same time the maximum energy which can be reached by the particle is reduced

proportionately by an amount $\Delta B\ 2B$, compared with the case where all sectors are identical.

(2) The second way consists in weakening the field in only a few magnets, but to a much more serious extent, for instance by making it zero or even negative there. As zero field magnets one can use quadrupole lenses which can have the same gradient as the main magnet units, for instance. Another variant is to replace some periods or parts of periods by equivalent magnetic structures [27]. These should include damping units with negative field and possess the same focusing properties as the corresponding elements of the main magnet system in order not to destroy its periodicity.

For constructional reasons it may be desirable to have the same field in all guiding magnets and to introduce special damping blocks in certain straight sections. Each of these blocks might consist of two magnets with opposite fields falling rapidly in absolute value with increasing distance from the centre of the accelerator, or just of a single magnet with negative field.

The method indicated under heading 2 may, under certain conditions, lead to a smaller decrease of the maximum electron energy than those under heading 1. However, they have their own disadvantages as they make the magnet system more intricate and some resonance bands broader.

Both methods reduce the damping of synchrotron oscillations. This is not desirable in itself as it is just these phase oscillations which are the most sensitive in terms of particle losses to the effect of quantum fluctuation of radiation (see below, § 5 and ch. 6).

Let us now pass to another method [13, 14, 24] which makes use of the coupling of radial and vertical betatron oscillations. The latter are damped in any periodic magnetic field without coupling of the x- and z-oscillations, as was seen in § 3, with a decrement $\langle\zeta_z\rangle = \frac{1}{2}\langle\Gamma\rangle$.

The principle of the method of using the x-z coupling is that one can reduce the oscillation of radial oscillations under certain conditions, at the cost of the damping of the z-oscillations. To realize this, the working point has to be taken near a difference resonance line (ch. 3, § 2):

$$\nu_x - \nu_z = k \qquad (k \text{ integer}). \tag{4.2}$$

The linearized equations of the x- and z-motion in the presence of radiation and compensating field, taking into account the coupling of co-ordinates and velocities, may be written as follows, according to (2.8), (2.9) and to (2.2) of ch. 3

$$x'' + (\Gamma_1 + \Gamma)x' + g_x x + h_{xz} z + h_\vartheta z' = \frac{K}{K_0^2}\,\delta\,, \tag{4.3}$$

$$z'' + (\Gamma_1 + \Gamma)z' + g_z z + h_{zx} x - h_\vartheta x' = 0\,, \tag{4.4}$$

where $\delta = \Delta E/E$ and where the coupling coefficients, which we assume to be small, are

$$h_{xz} = -\frac{e}{pcK_0^2}\frac{\partial b_z}{\partial z}\,, \qquad h_{zx} = \frac{e}{pcK_0^2}\frac{\partial b_x(1+Kx)}{\partial x}\,, \qquad h_\vartheta = \frac{eb_\vartheta}{pcK_0}\,. \tag{4.5}$$

The coupling defined by parameters h_{xz}, h_{zx} is realized either by using quadrupole lenses or by tilting the guide field magnet in a plane perpendicular to the orbit. The coupling

$$X = X(\nu, \varkappa, \delta, \varkappa, z, z) \tag{4.6}$$

and write the equations of motion (4.3), (4.4) and (2.17) in vector form. The matrix of the

but also by the coupling of the oscillations.

Let us look for a solution of the vector equation in the form

$$X = \sum a_j f_j e^{i(\nu_j \cdot \omega t} + \text{c.c.} \tag{4.7}$$

in the scheme set up in appendix D

$$\begin{aligned} \cdot &= \cdot \cdot \cdot

which indicates to what accuracy the working point should be kept at resonance and how large the coupling coefficients h_{xz}, h_{zx} and h_{ϑ} have to be.

The method making use of the coupling between x- and z-oscillations is advantageous because it does not reduce the average magnetic field on the orbit or the damping of the synchrotron oscillations. For accelerators with scaling orbits, where the sign of the decrement does not depend on the law $K = K(\vartheta)$ and is determined only by the quantity n_0, this method is apparently the only one which makes it possible to damp radial betatron oscillations (see ch. 7, § 3). In the case of strong-focusing synchrotrons (with non-scaling orbits) the method to be applied can only be decided on the basis of the practical values of the maximum electron energy, the length of the acceleration cycle and other parameters.

To conclude let us mention that it is impossible to damp radial oscillations by modifying the high-frequency accelerating system. As was shown in ch. 4, §§ 3 and 5, the decrements of the various types of oscillations (betatron, synchrotron) do not depend on the way the high-frequency accelerating system is realized.

5. EFFECT OF QUANTUM FLUCTUATIONS ON THE ELECTRON MOTION IN MAGNETIC SYSTEMS

As was shown in § 1, to take into account quantum effects when studying the motion of particles in an accelerator, one must start from the classical theory of motion of the radiating electron and include the corrections caused by the statistical fluctuations of the radiation. The importance of quantum effects in large electron accelerators was pointed out for the first time in [10] and [11].

To calculate the quantum corrections we shall in future make the following assumptions which are correct when $E \ll E_{\hbar}$ (see 1.11).

(a) The emissions of individual photons are statistically independent. The fact that the correlation corrections are not essential appears for instance from the expressions for W, ω_c (see 1.1, 1.4) according to which $\delta W/W \approx \hbar\omega_c/E \ll 1$.

(b) The photons are emitted instantaneously. The uncertainty in the time of emission has the order $\delta t \propto 1/\omega_c \propto 1/\omega\gamma^3$ and is a negligible quantity when compared to any characteristic time scale in our problem (period of oscillations, etc.).

(c) We shall make use of the expressions for the radiation spectrum and intensity obtained by the classical theory and correct to an accuracy up to terms of the order $(E/E_{\hbar})^2$.

The legitimacy of the treatment under assumptions (a), (b) and (c) in the region $E \ll E_{\hbar}$, that is for $\hbar\omega_c \ll E$, follows from the general statements

of radiation theory in quantum electrodynamics (see for instance [28, 29]).

To give an example, in a 5 GeV accelerator the electron at its maximum energy radiates on the average per turn about 100 photons $\hbar\omega_c$ having a wave-length $\lambda_c \approx 1$ Å (the orbit radius being about 15 m). The ratio $E/E_\hbar$ is then around 10^{-3}.

Let us derive the mean square amplitude value of betatron oscillations which are excited by the fluctuations of the radiation. Let an electron emit a photon with energy ε at some azimuth ϑ. This, according to (3.5), causes an instantaneous variation in the position of the equilibrium orbit and its slope at the point where it occurs, as well as an instantaneous change of the same features of the radial betatron oscillations

$$\begin{aligned} \delta y &= \frac{\psi(\vartheta)}{K_0}\frac{\varepsilon}{E}, \\ \delta y' &= \frac{\psi'(\vartheta)}{K_0}\frac{\varepsilon}{E}. \end{aligned} \tag{5.1}$$

As at any moment betatron oscillations have the form

$$y = a\,e^{i\nu\vartheta} f(\vartheta) + \text{c.c.}\,; \qquad y' = (f' + i\nu f) a\,e^{i\nu\vartheta} + \text{c.c.}\,, \tag{5.2}$$

their amplitude

$$A = 2|a|\,|f|_{\max} = |f|_{\max} |y(f^{*\prime} - i\nu f^*) - y' f^*| \tag{5.3}$$

experiences at the instant of emission an increase δA, which is given, according to (5.3), with an accuracy to terms of the order $(\delta y)^2$ and $(\delta y')^2$ inclusive, by the relation

$$\begin{aligned} \delta A^2 &= |f|^2_{\max}\{2y|f' + i\nu f|^2 \delta y + 2|f|^2 y' \delta y' - |f|^{2\prime}(y\delta y' + y'\delta y) + \\ &\quad + |f' + i\nu f|^2 (\delta y)^2 + |f|^2 (\delta y')^2 - |f|^{2\prime} \delta y \delta y' + \ldots\} = \\ &= \frac{|f|^2_{\max}}{K_0}\{2y\,|f' + i\nu f|^2 \psi + 2|f|^2 y'\psi' - |f|^{2\prime}(y\psi' + \psi' y)\}\frac{\varepsilon}{E} + \\ &\quad + \frac{|f|^2_{\max}}{K_0^2}\left\{|f' + i\nu f|^2 \psi^2 + |f|^2 \psi'^2 - |f|^{2\prime}\frac{\psi^{2\prime}}{2}\right\}\frac{\varepsilon^2}{E^2} + \ldots \end{aligned} \tag{5.4}$$

If one multiplies this expression by $\bar{n}_j$ – the average number of photons

with an energy ε_j emitted per unit of azimuth – and sums over all indices j, the quantity obtained will represent the average rate of increase of the square of the amplitude on account of the fluctuation. We notice that the time average of the first curly brackets in (5.4) gives zero because of the rapidly oscillating factors $y(\vartheta)$ and $y'(\vartheta)$. Physically this corresponds to the fact that to the first order in ε the photon emission does not lead on the average to an increase of the oscillations as it takes place with equal probability at all phases. To calculate the mean value of the second curly brackets, let us first note that with the assumptions made above

$$\varepsilon_j \bar{n}_j \approx \frac{U(u)}{cK_0}\,\mathrm{d}u$$

and (5.5)

$$\sum_j \varepsilon_j^2 \bar{n}_j \approx \int_0^\infty \frac{\hbar\omega_c}{cK_0}\, uU(u)\mathrm{d}u\ ; \qquad u = \frac{\omega}{\omega_c},$$

where $\hbar$ is Planck's constant, ω the photon frequency, ω_c the characteristic frequency of the spectrum (1.4), and $U(u)$ the spectral density of the radiation of the relativistic electron, travelling along an orbit with curvature K (see for instance [6, 8]):

$$U(u) = \frac{3\sqrt{3}}{4\pi}\, e^2 cK^2\gamma^4 u \int_u^\infty K_{\frac{5}{3}}(u')\mathrm{d}u'\,. \tag{5.6}$$

In this formula $K_{\frac{5}{3}}(u)$ is the Macdonald function of order $\frac{5}{3}$. We introduce (5.6) into (5.3), integrate by parts, and making use of the asymptotic and recurrent properties of the functions K_n (see [31, 32]), we arrive at the following expression

$$\sum_j \varepsilon_j^2 \bar{n}_j = \frac{55}{24\sqrt{3}}\,\frac{e^2\hbar cK^3}{K_0}\,\gamma^7\,. \tag{5.7}$$

Also, taking the mean value of (5.4) with respect to ϑ and to the photon spectrum, one arrives at following formula for the mean rate of increase of the amplitude squared because of quantum fluctuations

$$\left(\overline{\frac{\mathrm{d}A_x^2}{\mathrm{d}\vartheta}}\right)_{\text{mean quadratic value}} = \frac{55}{24\sqrt{3}}\, r_e \Delta F \gamma^5\,, \tag{5.8}$$

where $\Lambda = \hbar/m_0 c$ is the Compton wave-length, and $r_e = e^2/m_0 c^2$ the classical electron radius. The factor F is made up as follows:

$$F = |f_x|^2_{\max} \left\langle \frac{K^3}{K_0^3} \{ |f_x' + i\nu_x f_x|^2 \psi^2 + |f_x|^2 \psi'^2 - |f_x|^{2\prime} \psi\psi' \} \right\rangle \equiv$$

$$\equiv |f_x|^2_{\max} \left\langle \frac{K^3}{K_0^3} |(f_x' + i\nu_x f_x)\psi - f_x \psi'|^2 \right\rangle \qquad (5.9)$$

where one must take the average over a period of the system. For a magnet with rotational symmetry having sectors the factor F takes the form

$$F = \frac{|f_x|^2_{\max}}{(1-n)^2} \langle |f_x' + i\nu_x f_x|^2_B \rangle , \qquad (5.10)$$

where function f_x is determined by (2.44) and (2.45) of ch. 7, and the sign $\langle |\ |^2_B \rangle$ indicates the mean square value of the expression between the vertical lines taken between the limits of a section with magnetic field.

For the azimuthally symmetrical accelerator we have

$$F = \frac{1}{(1-n)^2} . \qquad (5.11)$$

The total variation of the quantity $\overline{A_x^2}$ is made up of two parts: the systematic variation due to the damping, and the part due to the fluctuation (5.8). The equation taking both into account has the form

$$\frac{d}{d\vartheta} \overline{A_x^2} = \frac{55}{24\sqrt{3}} r_e \Lambda F \gamma^5 - \overline{A_x^2} \left[2\langle \zeta_x \rangle + \frac{d \ln \gamma}{d\vartheta} \right] . \qquad (5.12)$$

A stricter derivation of this expression, taking into account also the simultaneous excitation of synchrotron oscillations by radiation fluctuations is given in appendix E. The complete expression for the mean square of the amplitude* of radial betatron oscillations considering the emission of radiation has, according to (5.12), the form

$$\overline{A_x^2} = A_{xi}^2 \frac{\gamma_i}{\gamma} \exp\left[-2 \int_0^{\vartheta} \zeta_x d\vartheta \right] + \frac{55}{24\sqrt{3}} \frac{r_e \Lambda F}{\gamma} \int_0^{\vartheta} \gamma^6 \exp\left[-2 \int_{\vartheta'}^{\vartheta} \zeta_x d\vartheta'' \right] d\vartheta' \qquad (5.13)$$

* Without considering particle losses at the vacuum chamber walls.

where A_{xi} is the initial value of the oscillation amplitude. The first term in (5.13) corresponds to the 'classical' component of the betatron oscillations and the second to the component due to the fluctuations. In general the behaviour of both components depends essentially on the sign and value of the decrement $\langle\zeta_x\rangle$. If $\langle\zeta_x\rangle < 0$, both parts in (5.13) are subject to the growth studied above (see § 3), which is, as we saw, of a purely classical nature. Usually the classical part of (5.13) exceeds the part due to fluctuations.

Methods of suppressing the build-up, and leading to the condition $\langle\zeta_x\rangle > 0$ have been considered above, in § 4. If this condition is satisfied, eq. (5.13) describes how the oscillation amplitude builds up under the action of the fluctuations and in the presence of damping. For the steady-state value $\overline{(A_x^2)}_{st}$ we find an expression which is valid at high energies, or rather when $\exp\int_0^{\vartheta}\zeta_x d\vartheta \gg 1$:

$$\overline{(A_x^2)}_{st} = \frac{55\sqrt{3}}{48}\frac{\Lambda F\gamma^2}{K_0 G_x}, \tag{5.14}$$

where G_x is the number given in (3.27) which characterizes the decrement $\langle\zeta_x\rangle$. For the azimuthally symmetrical accelerator this expression takes the following simple form

$$\overline{(A_x^2)}_{st} = \frac{55\sqrt{3}}{48}\frac{\Lambda\gamma^2}{K_0 n(1-n)}; \qquad 0 < n < 1. \tag{5.15}$$

If the energy or the acceleration time are sufficiently small, so that the radiation 'friction' cannot develop, we obtain from (5.13) the formula

$$\overline{A_x^2} = A_{xi}^2\frac{\gamma_i}{\gamma} + \frac{55}{24\sqrt{3}}\frac{r_e\Lambda F}{\gamma}\int_0^{\vartheta}\gamma^6(\vartheta')d\vartheta'. \tag{5.16}$$

In connection with formulae (5.14) to (5.16) one may make some general remarks. Because of the statistical character of the build-up process due to radiation quantum fluctuations, the amplitude, in the absence of damping, would grow in the average as the square root of the time. One may compare qualitatively the situation here to the oscillations of a pendulum (betatron oscillations), whose point of suspension (the instantaneous closed orbit) is subject to the action of random jerks

having a determined distribution. The process of build-up of oscillations can, in principle, continue for any length of time; taking into account the usual adiabatic damping $E^{-\frac{1}{2}}$ does not change the picture in practice and the characteristic stochastic growth proportional to $t^{\frac{1}{2}}$ is conserved. However, the situation changes very much when the radiation 'friction' studied above (3.1 and 3.10) becomes effective as this limits the growth to a duration equal to the damping time $\tau_x \approx 1/\omega_s \langle \zeta_x \rangle$. As we shall see, the picture will be the same for the excitation and stabilization of the synchrotron oscillation by radiation quantum fluctuations.

To be able to make rough estimates, we have given in table 4 the values of the quantity $F_1 = n^2 F$ as a function of μ for a strong-focusing system. These values were computed on the basis of formula (5.9) and correspond to a period satisfying the conditions $\vartheta_1 = \vartheta_2$, $K_1 = K_2 = K_0$, $|n_1| = n_2 = n \gg 1$ where the indices 1 and 2 refer to sectors with $n < 0$ and $n > 0$. The formulae for the functions $f(\vartheta)$ and $\psi(\vartheta)$, which are necessary for our computation are given in ch. 7 § 2. According to the results

TABLE 4

Values of the factor $F_1 = n^2 F$

$\sqrt{\lvert n_1 \rvert \theta_n}$	0	0.65	0.90	1.15	1.20	1.25	1.40	1.45	1.60	1.70	1.80	1.87
μ	0	0.25	0.47	0.78	0.86	0.93	1.20	1.30	1.65	1.94	2.36	π
F_1	∞	1145	346	154	136	120	91	72	77	85	140	∞
$\sqrt{F_1}$	∞	33.8	18.6	12.4	11.6	10.9	9.5	8.5	8.8	9.2	11.8	∞

laid down in table 4, the quantity F_1 becomes infinite at the limits of the region of stability ($\cos \mu = \pm 1$), as one could have expected, and has its minimum at about the centre of the region. In practical cases one should take into account the variation of F which will be caused by the modification of the magnet system necessary to damp radial betatron oscillations (alteration of the magnetic field along the azimuth, etc.). Let us assume, for instance a damping system in which the damping

parameter G_x in (3.27) is equal to one, as for a weak-focusing accelerator with $n \approx \frac{1}{2}$. Then the quantities $(\overline{A_x^2})^{\frac{1}{2}}$ for weak and strong focusing differ by a factor proportional to $F^{\frac{1}{2}} = F_1^{\frac{1}{2}}/|n|$, which is about equal to $10/|n|$ for $\cos \mu = 0$. However, the value of $|n|$ may be made sufficiently large and $(\overline{A_x^2})^{\frac{1}{2}}_{\text{strong foc}}$ may be made very much smaller than $(\overline{A_x^2})^{\frac{1}{2}}_{\text{weak foc}}$.

To obtain quantitative estimates let us transform expression (5.13) for $\overline{A_x^2}$ into

$$(\overline{A_x^2})^{\frac{1}{2}} = \left(\frac{55}{24\sqrt{3}} r_e \Lambda F c K_0 T_{\text{acc}} \gamma_{\max}^5\right)^{\frac{1}{2}} J_x(\xi, C_x), \tag{5.17}$$

where $J_x(\xi, C_x)$, which is determined by the formula

$$J_x^2(\xi, C_x) = \frac{1}{s(\xi)} \int_0^{\xi} s^6(\xi') \exp\left(-C_x \int_{\xi'}^{\xi} s^3(\xi'')\mathrm{d}\xi''\right) \mathrm{d}\xi', \tag{5.18}$$

is a function of the variable

$$\xi = t/T_{\text{acc}} \qquad 0 \leqslant \xi \leqslant 1, \tag{5.19}$$

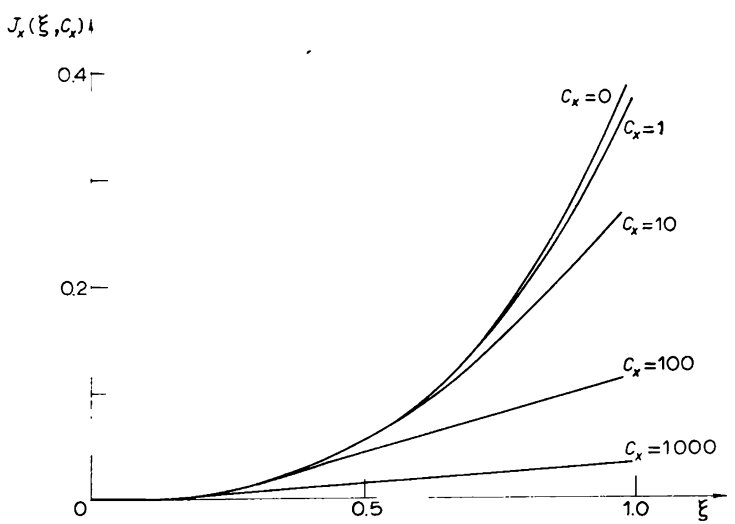

Fig. 46. The functions $J_x(\xi, C_x)$ for the case where the field depends linearly on time ($B \propto \xi$).

and of the constant C_x as a parameter

$$C_x = \tfrac{2}{3} r_e c K_0^2 G_x T_{acc} \gamma_{max}^3 \approx 4 \times 10^3 K_0^2 G_x T_{acc} E_{max}^3 , \tag{5.20}$$

where K_0 is in m^{-1}, T_{acc} in s, E_{max} in GeV.

The function $s(\xi)$ appearing in $J_x(\xi, C_x)$ characterizes the law of variation of the magnetic field with time. Typical cases are as follows:

(a) linear law $$s(\xi) = \xi , \tag{5.21}$$

(b) sinusoidal law $$s(\xi) = \sin \tfrac{1}{2}\pi\xi , \tag{5.22}$$

(c) superposition of a constant component and a sine-wave (variable magnetic field with full constant bias)

$$s(\xi) = \sin^2 \tfrac{1}{2}\pi\xi , \tag{5.23}$$

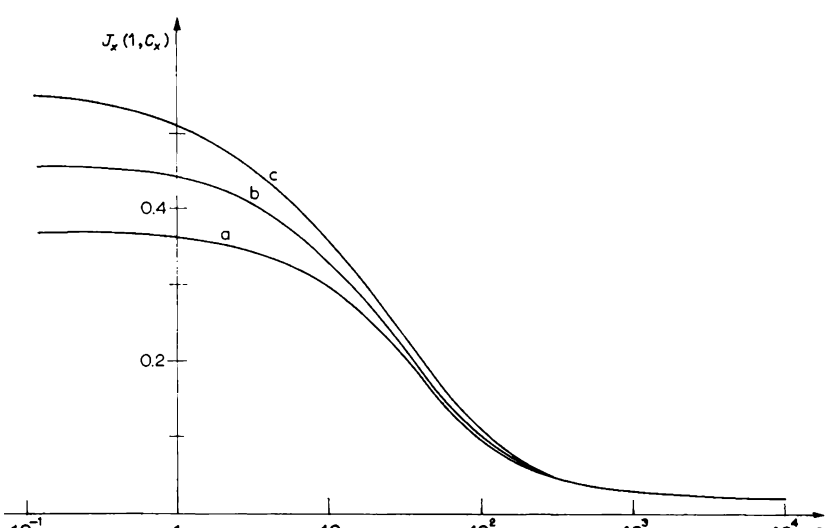

Fig. 47. Dependence of the quantity $J_x(1, C_x)$ on C_x for three laws of variation of the field with time: (a) linear; (b) sinusoidal; (c) sinusoidal with full bias.

In fig. 46 function $J_x(\xi)$ is given for the case $s(\xi)=\xi$ for different values of C_x. We observe further that for a given value C_x the values of $J_x(1, C_x)$ corresponding to the end of the acceleration cycle do not differ very much for the three cases (5.21) to (5.23). This can be seen in fig. 47, in which $J_x(1, C_x)$ is represented as a function of C_x for the three given functions $s(\xi)$.

Figures 48 and 49 give an idea of the quantity $(\overline{A_x^2})^{\frac{1}{2}}$ itself. In fig. 48 graphs of $(\overline{A_x^2})^{\frac{1}{2}}$ as a function of $\xi = t/T_{acc}$ are shown for a weak-focusing machine. Curves corresponding to the two boundary cases $(\overline{A_x^2})^{\frac{1}{2}}_{initial}$, $(\overline{A_x^2})^{\frac{1}{2}}_{stat}$ are drawn in this figure and marked I and II. These curves show to what extent the classical radiation damping of betatron oscillations considered in § 3 is effective. For experimental confirmation of the formulae derived above, see for instance ref. [36].

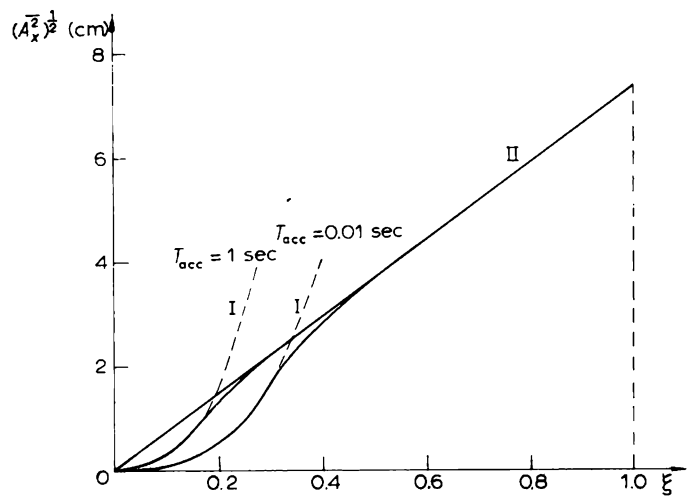

Fig. 48. Dependence of $(\overline{A_x^2})^{\frac{1}{2}}$ on the dimensionless time $\xi = t/T_{acc}$ in the case of weak focusing for $E_{max} = 5$ GeV, $n = \frac{2}{3}$, $K_0^{-1} = R = 15$ m.

The same dependence for a strong-focusing machine is shown in fig. 49, where it is assumed that $G_x \approx 2$. Approximate curves corresponding to $(\overline{A_x^2})^{\frac{1}{2}}_{initial}$ and $(\overline{A_x^2})^{\frac{1}{2}}_{stat}$ are also presented. For $n \gtrsim 100$ and $E_{max} \approx$ 5 GeV the quantity $(\overline{A_x^2})^{\frac{1}{2}}$ does not exceed a centimetre.

As we see, the quantity $(\overline{A_x^2})^{\frac{1}{2}}$ happens to be relatively small, i.e. by

an order of magnitude smaller than the radial dimensions of the aperture of the vacuum chamber. Of course, even for small values of $(A_x^2)^{\frac{1}{2}}$ there is some probability of losing particles by hitting the chamber walls. One should take into account these losses if the acceleration time is large. However, the pulse repetition rate in electron accelerators is

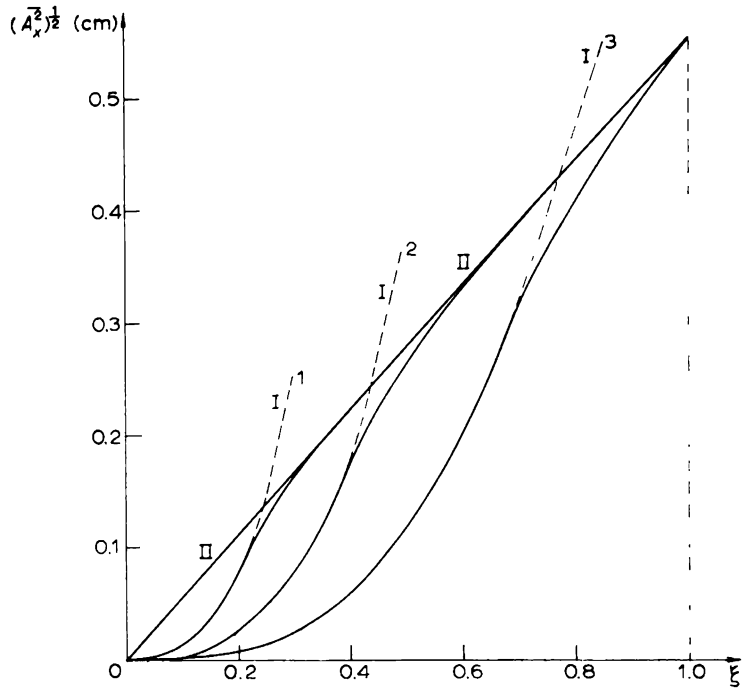

Fig. 49. Dependence of $(\overline{A_x^2})^{\frac{1}{2}}$ on the dimensionless time $\xi = t\ T_{acc}$ in the case of strong focusing for $E_{max} = 5$ GeV, $|n| = 100$, $K_0^{-1} = R = 15$ m, $\mu_x = \frac{1}{2}\pi$, $G_x \approx 2$, $\Lambda = 1/R\theta_M = 0.3$. Curve 1: $T_{acc} = 1$ sec; curve 2: $T_{acc} = 0.1$ sec; curve 3: $T_{acc} = 0.01$ sec.

usually chosen rather high (10 to 100 c/s) in order to increase the beam intensity so the acceleration time is small. Estimates show that with such small acceleration times the excitation of radial oscillations by radiation fluctuations cannot, in practice, lead to particle losses (see ch. 6).

The greatest danger is the excitation of phase synchrotron oscillations by radiation fluctuations. We now proceed to examine this.

One can obtain an expression for the mean square value of the amplitude A_η^2 by the same considerations as above. Upon the emission of a quantum the deviation of the phase from its equilibrium value does not change. It is only the quantity $\Delta E/E$ which experiences a jump, and consequently also η':

$$\delta\eta = 0\,, \qquad \delta\eta' = -q\alpha\,\frac{\varepsilon}{E}\,, \tag{5.24}$$

where ε is the photon energy and α the coefficient of orbit expansion. Thus, the increase of the square of the amplitude becomes

$$\delta A_\eta^2 = -\frac{2q\alpha}{\nu_s^2}\,\frac{\varepsilon}{E}\,\eta' + \frac{q^2\alpha^2}{\nu_s^2}\,\frac{\varepsilon^2}{E^2}\,. \tag{5.25}$$

Multiplying both parts of (5.25) by $\bar{\eta}_j$ and summing over all indices j, we obtain by analogy with (5.8)

$$\frac{\mathrm{d}\overline{A_\eta^2}}{\mathrm{d}\vartheta} = \frac{55\pi}{12\sqrt{3}}\,\frac{\alpha q e \Lambda \langle K^3\rangle}{K_0 V \sin\varphi_s}\,\gamma^6 - \overline{A_\eta^2}\left[2\langle\zeta_s\rangle + \frac{1}{2}\,\frac{\mathrm{d}\ln\gamma V\sin\varphi_s}{\mathrm{d}\vartheta}\right] \tag{5.26}$$

where the second part on the right-hand side describes the damping of the oscillations due to the change of the parameters and radiation effects considered in § 3. Integration of eq. 5.26 gives (with zero initial conditions)

$$\overline{A_\eta^2} = \frac{55\pi}{12\sqrt{3}}\,e\Lambda q\alpha\,\frac{\langle K^3\rangle}{K_0}\,(V\sin\varphi_s)^{-0.5}\gamma^{-0.5}\int_0^{\vartheta}\gamma^{6.5}(V\sin\varphi_s)^{-0.5}\times$$

$$\times\exp\left(-2\int_{\vartheta'}^{\vartheta}\langle\zeta_s(\vartheta'')\rangle\,\mathrm{d}\vartheta''\right)\mathrm{d}\vartheta'\,. \tag{5.27}$$

Let us consider again two boundary cases: the initial stage when the radiation friction, described by the function $\exp(-2\int_{\vartheta'}^{\vartheta}\langle\zeta_s(\vartheta'')\rangle\,\mathrm{d}\vartheta'')$, is

not yet felt, and the more important second stage, when this friction sets a limit to the growth of the amplitude of the oscillations excited by quantum fluctuations. In the initial stage the amplitude $(\overline{A_\eta^2})_{\text{init}}$ is described by the same formula (5.27) dropping the exponential factor under the integral. The stationary value of the amplitude corresponding to the second stage and determined by the electron energy is proportional to $(\overline{A_\eta^2})_{\text{stat}} \propto (V^{-1}\gamma^6/2\langle\zeta_s\rangle)$. In the region of high energies in which we are interested, practically all the power of the high-frequency accelerating system is used up in compensating the radiation losses (see § 1)

$$eV\cos\varphi_s \approx \frac{4\pi}{3}\frac{e^2\gamma^4}{K_0}\langle K^2\rangle\,. \tag{5.28}$$

Making use of this, we get from (5.27)

$$(\overline{A_\eta^2})_{\text{stat}} \approx \frac{55\sqrt{3}}{32}\frac{\Lambda}{r_e}\frac{q\alpha}{K_0}\frac{\langle K^3\rangle}{\langle K^2\rangle}\frac{\operatorname{ctg}\varphi_s}{G_s}\frac{1}{\gamma}\,. \tag{5.29}$$

Formula (5.29) becomes much simpler in the case of the azimuthally symmetrical magnet with straight sections and takes the form

$$(\overline{A_\eta^2})_{\text{stat}} \approx \frac{55\sqrt{3}}{32}\frac{\Lambda}{r_e}\frac{q\operatorname{ctg}\varphi_s}{(1+\lambda)(3-4n)}\frac{1}{\gamma}\,, \tag{5.30}$$

where $\lambda = L/2\pi R$, L being the total length of all straight sections and R the orbit radius in the magnet sectors.

Expression (5.27) for $\overline{A_\eta^2}$ can be written in a form analogous to (5.17) and convenient for fast quantitative estimates. For this, however, one has to make practical assumptions as regards the dependence of the quantities V and φ_s on energy (or on time).

If we make the extreme hypothesis

$$\varphi_s = \text{const.}\,, \qquad V = V_{\max}\left(\frac{\gamma}{\gamma_{\max}}\right)^4, \tag{5.31}$$

then (5.27) may be written as

$$(\overline{A_\eta^2})^{\frac{1}{2}} = \left(\frac{55\pi}{12\sqrt{3}}\frac{ce\Lambda q\alpha\langle K^3\rangle T_{\text{acc}}\gamma_{\max}^6}{V_{\max}\sin\varphi_s}\right)^{\frac{1}{2}} J_\eta(\xi, C_\eta)\,, \tag{5.32}$$

where $$J_\eta^2(\xi, C_\eta) = \frac{1}{s^{2.5}(\xi)} \int_0^\xi s^{4.5}(\xi') \exp\left(-C_\eta \int_{\xi'}^\xi s^3(\xi'')\mathrm{d}\xi''\right)\mathrm{d}\xi' . \quad (5.33)$$

The quantities $s(\xi)$ and ξ have the same meaning as before (5.21 to 5.23, and 5.19), whereas the constant C_η is equal to

$$C_\eta = \tfrac{2}{3} r_e c K_0^2 G_s T_{acc} \gamma_{max}^3 \approx 4 \times 10^3 K_0^2 G_s T_{acc} E_{max}^3$$

where K_0 is in m^{-1}, T_{acc} in sec, E_{max} in GeV.

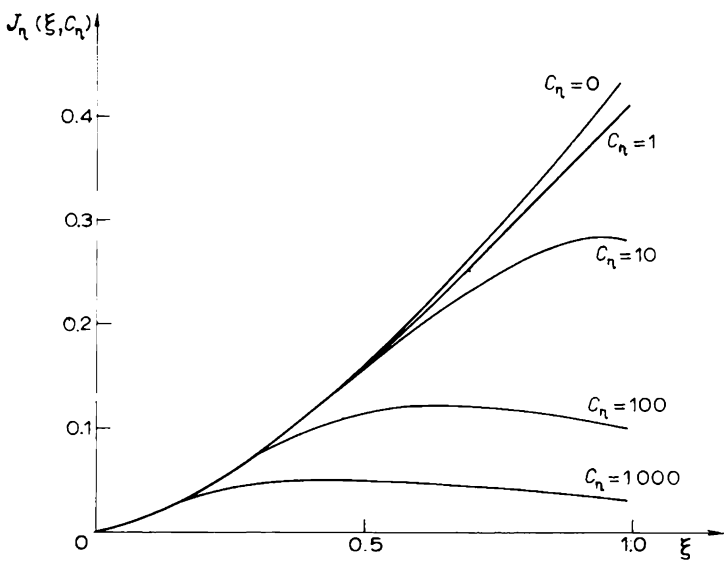

Fig. 50. The functions $J_\eta(\xi, C_\eta)$ for the case where the field depends linearly on time ($B \sim \xi$).

Figure 50 shows function $J_\eta(\xi)$ for various C_η, the function $s(\xi)$ being given by eq. (5.21). In fig. 51 the values of $J_x(1, C_x)$, corresponding to the end of the acceleration cycle, are plotted for the three functions $s(\xi)$ given in (5.21) to (5.23).

One can estimate the order of magnitude of the quantity $(\overline{A_\eta^2})^{\frac{1}{2}}$ itself with figs. 52 and 53 where $(\overline{A_\eta^2})^{\frac{1}{2}}$ is drawn as a function of $\xi = t/T_{acc}$ during the acceleration cycle in the various cases. One sees that the amplitude of the phase oscillations excited by the fluctuations is rather

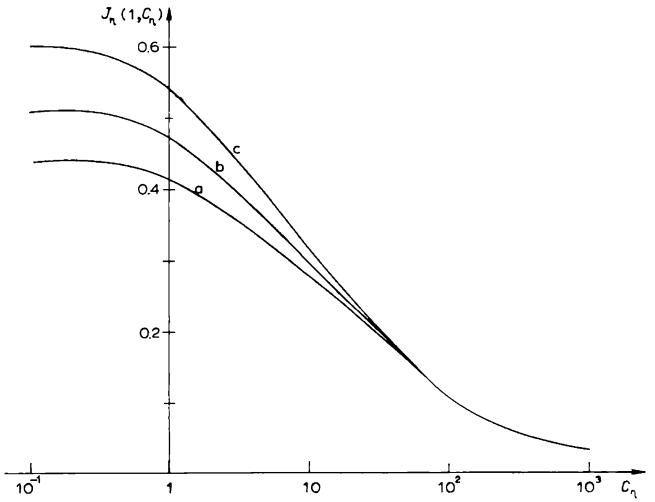

Fig. 51. Dependence of the quantity $J_\eta(1, C_\eta)$ on C_η for three laws of variation of the field with time: (a) linear; (b) sinusoidal; (c) sinusoidal with full bias.

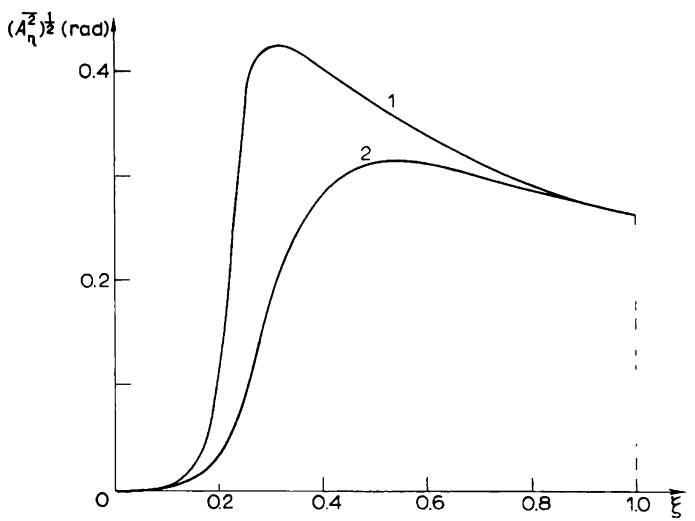

Fig. 52. Dependence of $(\overline{A_\eta^2})^{\frac{1}{2}}$ on the dimensionless time $\xi = t\ T_{acc}$ in the case of weak focusing for $E_{max} = 5$ GeV, $n = \frac{2}{3}$, $R = 15$ m, $q = 1$, $\varphi_s = 60°$.
Curve 1: $T_{acc} = 1$ sec; curve 2: $T_{acc} = 0.1$ sec.

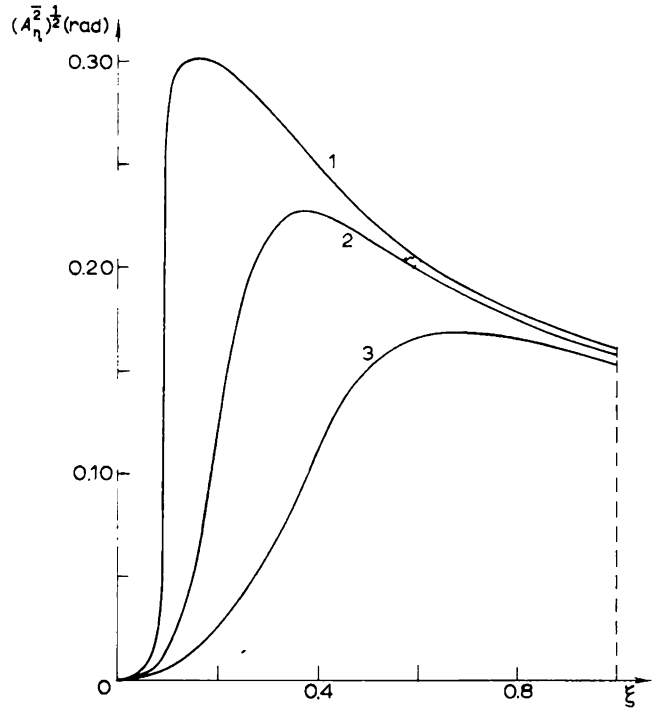

Fig. 53. Dependence of $(\overline{A_\eta^2})^{\frac{1}{2}}$ on the dimensionless time $\xi = t/T_{acc}$ in the case of strong focusing for $E_{max} = 5$ GeV, $|n| = 100$, $l/R\theta_M = 0.3$, $\mu_x = \frac{1}{2}\pi$, $q = 100$, $\varphi_s = 60°$. Curve 1: $T_{acc} = 1$ sec; curve 2: $T_{acc} = 0.1$ sec; curve 3: $T_{acc} = 0.01$ sec.

large and decreases relatively slowly in the region of high energies (as $E^{-\frac{1}{2}}$, see 5.27, 5.30), so that the azimuthal dimensions of the bunch of accelerated particles become important. Because of this, in particular, a coherent emission of radiation, as mentioned in § 1, cannot take place in practice, even at high energies [17].

As appears from (5.27), and also from figs. 52 and 53, the quantity $(\overline{A_\eta^2})^{\frac{1}{2}}$ increases at the beginning. However, its maximum value is not necessarily reached at the end of the acceleration. Depending on the length of the acceleration and on other parameters, it can be reached long before this moment. In such a case $(\overline{A_\eta^2})^{\frac{1}{2}}$ goes through a maximum and decreases then according to (5.29).

Calculations show that in a weak-focusing synchrotron for an energy of 1.5 to 2 GeV, the excitation of phase oscillations by radiation fluctuations represents a real danger and can lead to loss of particles. This is due to the fact that the value of $(\overline{A_\eta^2})^{\frac{1}{2}}$ can become comparable to the bucket dimensions although, as was mentioned earlier in connection with $(\overline{A_x^2})^{\frac{1}{2}}$, the acceleration time is also essential for particle losses. A further study of this question and the calculation of the loss of particles are presented in the next chapter.

Besides oscillations of the energy Δt and the phase η, synchrotron oscillations of the radial position x_s are also excited. The maximum displacement x_s, which plays the role of the amplitude A_{xs}, is equal to

$$A_{xs} = \frac{\psi_m}{K_0} \frac{\Delta E}{E_s}, \tag{5.34a}$$

where ψ_m is the maximum value of function ψ. According to (2.19) the quantity $\overline{A_{xs}^2}$ may be expressed in the following manner

$$\overline{A_{xs}^2} = \frac{\psi_m^2 \nu_s^2}{q^2 K_0^2 \alpha^2} \overline{A_\eta^2} = \frac{r_e \psi_m^2 V \sin \varphi_s}{2\pi e q \alpha K_0^2 \gamma} \overline{A_\eta^2}, \tag{5.35}$$

$$\overline{A_{xs}^2} \approx \frac{55\sqrt{3}}{48} \frac{\Lambda \psi_m^2}{K_0^4} \frac{\langle K^3 \rangle \gamma^2}{G_s}, \tag{5.36}$$

where G_s is the damping parameter (3.28). The radial synchrotron oscillations excited by the fluctuations are small in absolute value, like the radial betatron oscillations examined earlier, and have amplitudes of the order of 1 cm, for $|n| \gtrsim 100$ and even up to an energy $E \lesssim 10$ GeV.

In connection with the results obtained concerning the excitation of phase and radial synchrotron oscillations, a few remarks are appropriate on the choice of the parameters for the accelerating system, in particular the harmonic number. With a harmonic number equal to q, one finds on the orbit q regions of stability buckets, the azimuthal dimensions of which are inversely proportional to the value of q. On the other hand the amplitude of the phase oscillations excited by radiation fluctuations is proportional to the square root of the harmonic number $(\overline{A_\eta^2})^{\frac{1}{2}} \propto \sqrt{q}$. For this reason, there is in each practical case a limit q_{max} set to the harmonic number for which the quantity $(\overline{A_\eta^2})^{\frac{1}{2}}$ becomes comparable with the bucket dimensions during acceleration, so that there are losses

due to particles falling out of these regions of stability. Computations made with formulae (5.27) and (5.29) in the case of weak focusing with reasonable parameters show that $q_{\max}$ comes down to unity already in accelerators designed for an energy $E_{\max} \gtrsim 1.5$ to 2 GeV. Thus, acceleration in a weak-focusing synchrotron up to high energies is made more difficult because of the excitation of phase oscillations by radiation fluctuations. In a strong-focusing machine the situation is much more favourable. This is in general due to the fact that $(A_\eta^2)^{\frac{1}{2}} \propto q\alpha$, where the value of α is considerably smaller in this case (for instance, by two orders of magnitude) than for weak focusing, where $\alpha = 1/(1-n)$. So the excitation of phase oscillations by the fluctuations is not, in a strong-focusing accelerator, an obstacle to the choice of a very high harmonic number, even of the order of several hundred (see for instance [33, 34]). This feature is essential, as there are a number of reasons why the harmonic number must have a sufficiently large value in such an accelerator. So, because of resonances which are dangerous for betatron oscillations, the part of the annular region of the magnetic field where instantaneous orbits may be is rather small in terms of radii. It is thus necessary, in order not to lose beam intensity, to ensure that the radial extension of the bucket remains small, even if V is large, and to choose a large q. Also, for the radiofrequency-system, it may be convenient to go to relatively short wave-lengths (say of the order of one metre) and consequently to large q-values.

Up to now we have considered only the motion in the plane $z = 0$. Let us now examine the z-oscillations (see refs. [12–14]), whose excitation by radiation fluctuations differs from that of the phase and radial oscillations we have considered above. Let us keep in mind that the photon with an energy $\varepsilon = \hbar\omega$ is not emitted exactly in the direction of the velocity of the electron, but with a slight angle $\chi \approx 1/\gamma$ on the average. Each time a photon is emitted at an angle χ the electron experiences a recoil in a direction perpendicular to its speed, which on the average will excite an oscillation. As far as the x-oscillations are concerned this effect gives only small corrections to the expression (5.13) obtained above, but for the z-oscillations the fluctuation acts mainly in their direction. However, as in the other cases, the radiation friction (3.1), (3.3) does not allow the amplitude A_z to attain a large value as it limits its increase to a length of time equal to the damping time $I \propto 1/\langle \Gamma \rangle \omega_s$.

The linearized equation of motion for z (2.9), taking into consideration radiation fluctuations, has the form

$$z''+(\Gamma_1+\Gamma)z'+g_z z=-\frac{1}{K_0}\sum_i \frac{\varepsilon_i}{E_s}\,\delta(\vartheta-\vartheta_i)\sin\chi_i\,. \tag{5.37}$$

Starting with the usual solution of the type

$$z=a_z e^{i\nu_z\vartheta}f(\vartheta)+\text{c.c.} \tag{5.38}$$

and using the method of variation of the constants, we obtain for a_z the expression

$$a_z=-\frac{1}{2iK_0}\int_0^{\vartheta} f_z^*\sum_i\frac{\varepsilon_i}{E_s}\,\delta(\vartheta'-\vartheta_i)\sin\chi_i\exp\left[-\int_{\vartheta'}^{\vartheta}\left(i\nu_z+\frac{\Gamma_1+\Gamma}{2}\right)d\vartheta''\right]d\vartheta'. \tag{5.39}$$

To find the mean square fluctuation of the vertical oscillation amplitude

$$\overline{A_z^2}=4|f_z|_m^2\langle\overline{|a_z|^2}\rangle\,, \tag{5.40}$$

one has to know the expression for $\kappa(k,\chi)$, the probability of emitting a photon of the kth harmonic at an angle χ to the velocity in a vertical plane. To begin with let us use the formula for the spectral radiation intensity of the kth harmonic (see 1.5). In the case $\gamma\gg 1$ this formula leads to an expression accurate up to terms of the order γ^{-4}:

$$\kappa(k,\chi)=\frac{k^2}{4\pi^3\gamma^4}\left[g^2K_{\frac{2}{3}}(\tfrac{1}{3}kg^{\frac{3}{2}})+g\sin^2\chi K_{\frac{1}{3}}(\tfrac{1}{3}kg^{\frac{3}{2}})\right]do\,. \tag{5.41}$$

Here $K_{\frac{2}{3}}, K_{\frac{1}{3}}$ are Macdonald functions of the order $\frac{2}{3}$ and $\frac{1}{3}$, do is the element of solid angle and

$$g=1-\beta^2\cos\chi\,. \tag{5.42}$$

If we take the average of $|a_z|^2$ over the photon spectrum, the angles of flight and the phases of the vertical betatron oscillations, we arrive at the following relations

$$\overline{A_z^2}=\frac{13}{24\sqrt{3}}\,\frac{r_e\Lambda}{\gamma}\,|f_z|_m^2\left\langle\frac{K}{K_0}\,|f_z|^2\right\rangle\int_0^{\vartheta}\gamma^4\exp\left(-\int_{\vartheta'}^{\vartheta}\Gamma d\vartheta''\right)d\vartheta' \tag{5.43}$$

$$(\overline{A_z^2})_{\text{stat}} = \frac{13\sqrt{3}}{48} \Lambda |f_z|_m^2 \frac{\langle K|f_z|^2\rangle}{\langle K^2\rangle}, \tag{5.44}$$

which give for an azimuthally symmetrical field

$$\overline{A_z^2} = \frac{13}{24\sqrt{3}} \frac{r_e \Lambda}{n\gamma} \int_0^{\vartheta} \gamma^4 \exp\left(-\int_{\vartheta'}^{\vartheta} \Gamma \mathrm{d}\vartheta''\right) \mathrm{d}\vartheta', \tag{5.45}$$

$$(\overline{A_z^2})_{\text{stat}} = \frac{13\sqrt{3}}{48} \frac{\Lambda}{Kn}. \tag{5.46}$$

It is interesting to note that the stationary value $(\overline{A_z^2})_{\text{stat}}$ which is established does not depend on energy. The physical reason for this is that although recoil momentum and number of photons increase with energy, the angle of flight χ decreases correspondingly, so that these factors compensate each other in the average. The amplitude of the excited oscillations $(\overline{A_z^2})^{\frac{1}{2}} \lesssim (N/K)^{\frac{1}{2}}$ is extremely small, so that the effect considered does not present any danger in practice.

The expressions obtained above for the mean square quantities $\overline{A_x^2}, \overline{A_\eta^2}$, etc., make it possible to compute the electron losses due to radiation effects. To such a calculation is devoted § 3 of the next chapter, where one will find the theory of particle losses by statistical effects for various conditions of damping.

CHAPTER 6

PARTICLE LOSSES CAUSED BY RANDOM PERTURBATIONS OF THE MOTION

As was shown in previous chapters, the motion of particles in an accelerator is subject to various perturbations, which can cause both the excursion of the closed orbit and the amplitude of betatron and synchrotron oscillations to increase to a dangerous value. This leads to loss of particles against the vacuum chamber walls and reduces the efficiency of the machine. As far as magnetic field perturbations and parameters of the accelerating system are concerned, this question has been partially examined in ch. 3 and 4.

However, there are also other phenomena which perturb the motion of particles, as for instance scattering by residual gas, radiation fluctuations (ch. 5), production of mesons and other secondary particles, etc. All these perturbations have in common a random time dependence which has an action analogous to that of noise in an accelerating system (see ch. 4 § 6). As we shall see, these effects can reduce the intensity of the beam so that knowledge of them is extremely important for correct design and operation of the accelerator.

One can divide random perturbations leading to loss of particles somewhat arbitrarily in two groups, small ones which, however, act during a long period, and the relatively large ones. To the first group belong small angle scattering by residual gas, quantum fluctuation of radiation, effect of noise on betatron and synchrotron oscillations, and multiple traversals through weak resonances. Problems of this type lend themselves to a general approach which will be explained in principle in the course of the present chapter. Losses of the second kind, for instance by scattering of particles at large angles or by passing through a strong resonance, are easier to estimate and we shall only dwell on them briefly.

So we shall be concerned in principle with the repeated action of small statistically independent perturbations in the course of a long period of time. We shall use the results of the preceding chapters, in which we calculated a few statistical characteristics of betatron and synchrotron oscillations excited by perturbations of this type. In this chapter we shall try to evaluate them from a somewhat different point of view and shall calculate the beam losses they cause.

1. BASIC EQUATIONS AND PRESENTATION OF THE PROBLEM

A problem of this type was solved for the first time in connection with an accelerator in paper [1], where the effect of multiple proton scattering by residual gas on betatron oscillations in a synchrophasotron was examined. Let us note that it is important in such a case to take the damping of the oscillations accurately into account, which is however difficult mathematically. Estimates show that even weak adiabatic damping may modify the value of the integrated scattering losses quite noticeably, while in the case of electron accelerators calculation of radiation fluctuation losses without considering radiation damping leads to severely overestimated or to simply incorrect figures.

In this chapter we shall examine the effect of perturbations on one of the types of oscillations without considering the remaining degrees of freedom. Without spending time on justifying this approach let us state only that it is valid in any case where the losses due to the excitation of any one of the oscillation types predominate, as is usually the case in practice. As was shown in previous chapters, one may write linear oscillations in any degree of freedom in the form

$$x(t) = D(t)\,\mathrm{Re}\,[Cf(t)\mathrm{e}^{\mathrm{i}\int\Omega\mathrm{d}t}]\,, \tag{1.1}$$

where $D(t)$ is a factor describing the damping of the oscillation, $f(t)$ the periodic Floquet function, and Ω the frequency of the kind of oscillation considered (for synchrotron motion one may write $f = \text{const.}$). The quantities C and C^* are, in the absence of perturbations, integrals of motion. We shall be mainly interested in the amplitude distribution of the particles, and not in their phase distribution, for which one must consider the variation with time of constant C produced by external perturbations. Because of the damping, the maximum permissible value

$|C|_{\max}$ increases with time as $D^{-1}(t)$. So it is convenient to take the ratio $y = |C|^2/|C|^2_{\max}$ as independent variable, $|C|^2_{\max}$ corresponding to some particular instant, as for instance $t = 0$ *. If we put $D(0) = 1$, which is always possible, the maximum permissible value of y at any moment t will be equal to $D^{-2}(t)$, showing up the effect of the damping.

The action of small statistical independent perturbations can be described by the Einstein–Fokker equation applied to the particle distribution function $Y(y, t)$:

$$\frac{\partial Y(y,t)}{\partial t} = -\frac{\partial}{\partial y}(\overline{\Delta y}\, Y(y,t)) + \frac{\partial^2}{\partial y^2}\left(\frac{\overline{(\Delta y)^2}}{2}\, Y(y,t)\right), \tag{1.2}$$

where $\overline{\Delta y}$ and $\overline{(\Delta y)^2}$ are the mean increase and the mean square of the increase of the quantity y per unit time**.

Physically eq. (1.2) is simply a continuity equation, as it can be written in the form

$$\frac{\partial Y}{\partial t} + \operatorname{div} I = 0\,,$$

where I is the particle current along axis y. As follows from (1.2), this current consists of two parts, a systematic component corresponding to the mean displacement velocity $\overline{\Delta y}$, and a diffusion component proportional to the gradient of concentration. An analogous equation describes for instance the diffusion process or Brownian motion.

Let us go over now to new variables

$$\begin{aligned} u &= D^2(t)\, y\,, \\ \tau &= \int_0^t D^2(t)\,\overline{\Delta y}(t)\,\mathrm{d}t\,, \\ U(u,\tau) &= \left[D^{-2}(t)\, Y(y,t)\right]_{\substack{y(u,\tau) \\ t(\tau)}}, \end{aligned} \tag{1.3}$$

* It should be noted that we have taken as independent variable the square of the amplitude, not the amplitude itself, that is a quantity canonically conjugate to the oscillation phase.

** The strict derivation of the Einstein–Fokker equation and the limits between which it can be applied are given in ref. [2]. A somewhat simplified derivation can also be found in [3] and [4].

which all have a simple physical sense. The quantity $u^{\frac{1}{2}}$ is evidently equal to the real oscillation amplitude expressed in terms of the maximum permissible amplitude. The dimensionless 'time' τ is numerically equal to the mean square of the amplitude of the oscillations which would have been excited at the time t by the stochastic process without damping. As the increases Δy and $(\overline{\Delta y})^2$ are linked, up to terms of the order $(\Delta C)^4$, by the relation

$$\overline{(\Delta y)^2} = 2y\overline{\Delta y}\,, \tag{1.4}$$

we obtain from (1.2), when $\overline{\Delta y}$ is independent of y,

$$\frac{\partial U}{\partial \tau} = \frac{\partial}{\partial u}\left[u\frac{\partial U}{\partial u} + \kappa^2(\tau)uU\right], \tag{1.5}$$

where the parameter $\kappa^2(\tau)$

$$\kappa^2(\tau) = -\frac{1}{D^4\overline{\Delta y}}\frac{\mathrm{d}D^2}{\mathrm{d}t} = \frac{y_{\max}}{\overline{\Delta y}}\frac{\mathrm{d}y_{\max}/\mathrm{d}t}{y_{\max}} \tag{1.6}$$

characterizes the effect of damping on the process of stochastic excitations of oscillations. One can show that κ^{-2} is the ratio of the mean square of the remaining amplitude to the maximum permissible value of the square of the amplitude. Hence, it is clear that the character of the diffusion process described by (1.5) depends essentially on the value of κ^2. If $\kappa^2 \gg 1$ the remaining amplitude is small, i.e. the damping is strong. If $\kappa^2 \ll 1$ the diffusion process takes place practically without damping and the losses for a similar interval of time are much larger.

Equation (1.5) must be supplemented by the boundary and initial conditions

$$U(1,\tau) = 0\,, \tag{1.7}$$

$$U(u,0) = Y_0(u) = Y_0(y)\,, \tag{1.8}$$

the first of which expresses the fact that particles with an amplitude equal to the maximum permissible are lost. The function $Y_0(y)$ characterizes the initial amplitude distribution, which depends on the practical conditions.

According to its definition the remaining amplitude does not depend

explicitly on time and can vary only when the energy or other parameters vary. In this connection one may consider the quantity κ^2 in eq. (1.5) as varying slowly in a number of cases, and sometimes simply as a constant (for instance in electron storage rings). So it becomes possible to solve eq. (1.5) by approximation methods.

After the distribution function $U(u, \tau)$ has been found, one can calculate the number of particles which are still being accelerated at time τ

$$N(\tau) = \int_0^1 U(u, \tau) \mathrm{d}u \,. \tag{1.9}$$

This makes it possible to formulate the requirements for the parameters of the accelerator in order to have sufficient beam intensity.

2. PARTICLE LOSSES IN THE ABSENCE OF DAMPING. MULTIPLE SCATTERING OF PROTONS BY RESIDUAL GAS

Although the case $\kappa^2 = 0$ (absence of damping) is a special case of the general problem examined below, it is appropriate to study it separately because it is relatively simple. In addition, it can be used in a number of cases, where κ^2 is small, as a rather good approximation to the practical conditions.

Equation (1.5) is easy to solve for $\kappa^2 = 0$ by separation of the variables whereby the solution is expressed in terms of Bessel functions of order zero

$$U = \sum_{i=1}^{\infty} C_i \mathrm{e}^{-\mu_i \tau} J_0(2\sqrt{\mu_i u}) \,, \tag{2.1}$$

the quantities μ_i being the eigenvalues of the problem, i.e. the roots of equation

$$J_0(2\sqrt{\mu_i}) = 0 \,, \tag{2.2}$$

and the constants C_i being determined by the initial conditions

$$C_i = J_1^{-2}(2\sqrt{\mu_i}) \int_0^1 Y_0(u) J_0(2\sqrt{\mu_i u}) \mathrm{d}u \,. \tag{2.3}$$

Making use of the solution obtained, we find from formula (1.9) the quantity of particles which are still being accelerated at instant τ

$$N(\tau) = \sum_{i=1}^{\infty} C_i e^{-\mu_i \tau} \int_0^1 J_0(2\sqrt{\mu_i u})\, du = \sum_{i=1}^{\infty} \frac{C_i}{\sqrt{\mu_i}} J_1(2\sqrt{\mu_i}) e^{-\mu_i \tau}. \quad (2.4)$$

Expression (2.4) is the solution to the problem given. In particular, if all particles had zero amplitude at the start, that is $Y_0(u) = \delta(u)$, one obtains

$$C_i = J_1^{-2}(2\sqrt{\mu_i}) \quad (2.5)$$

and

$$N(\tau) = \sum_{i=1}^{\infty} \frac{\exp(-\mu_i \tau)}{\sqrt{\mu_i} J_1(2\sqrt{\mu_i})}. \quad (2.6)$$

The number of remaining particles is plotted as a function of parameter τ in fig. 54, from which one sees that the quantity τ has to be

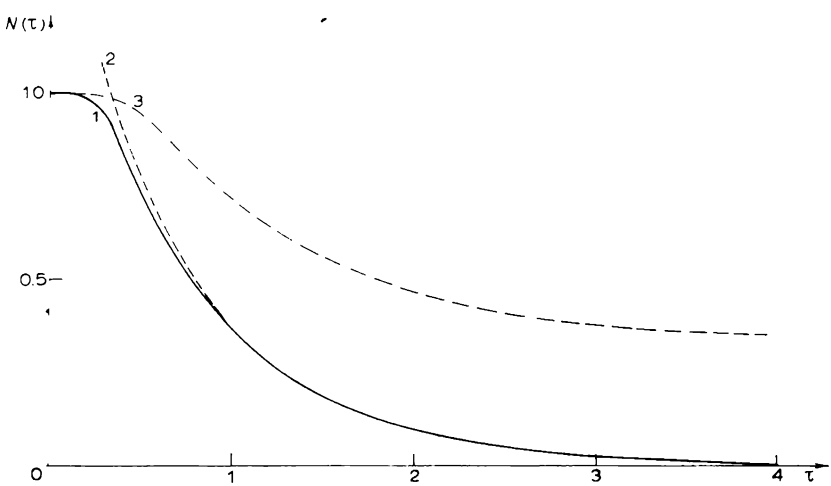

Fig. 54. Effect of small random perturbations of the motion on beam intensity in an accelerator. (1) without taking damping into account, (2) asymptotic curve (2.7), (3) lower limit of the losses taking into account adiabatic damping of betatron oscillations (non-relativistic case).

small, say of the order of 0.2 or 0.3, if the losses are to remain small. It should be remembered that in this case τ signifies the mean square oscillation amplitude at time t measured in terms of the permissible amplitude. This gives a simple criterion for estimating the tolerable perturbation value.

The eigenvalues μ_i of the problem considered increase rapidly with the index i (ref. [5]): $\mu_1 = 1.45$; $\mu_2 = 7.6$; $\mu_3 = 18.6$. So, for sufficiently high τ ($\tau \gg \mu_2^{-1} = 0.13$) in formula (2.6) one may neglect all terms except the first:

$$N(\tau) \approx \frac{e^{-\mu_1 \tau}}{\sqrt{\mu_1} J_1(2\sqrt{\mu_1})} = 1.60\, e^{-1.45\tau}. \tag{2.7}$$

The corresponding asymptotic curve is drawn dotted in fig. 54. One sees at once that formula (2.7) gives a sufficiently good approximation in the region where particle losses are substantial.

As an example, let us consider multiple elastic scattering of protons on residual gas in the synchrophasotron [1, 6]. Let us remark that one may limit oneself in the case of protons to the elastic scattering as the inelastic cross-section (including nuclear processes) is smaller by orders of magnitude in the energy range where scattering is large.

From the point of view of particle dynamics in an accelerator, elastic scattering is equivalent to an instantaneous change in the direction of the particle by the diffusion angle χ. Thus the integral y of the betatron oscillations changes by a quantity δy, whose mean value is

$$\overline{\delta y} = \frac{\overline{|\Delta C|^2} |f|^2_{\max}}{A^2_{\text{perm}}} = \frac{\overline{\chi^2} \langle |f|^2 \rangle |f|^2_{\max}}{A^2_{\text{perm}} K_0^2}. \tag{2.8}$$

Here A_{perm} is the permissible oscillation amplitude, $\overline{\chi^2}$ the mean square scattering angle, and f the Floquet function for the betatron oscillations in the direction considered (usually one examines the scattering in the vertical direction as the vacuum chamber dimensions are less than in the radial direction). One can get an idea of the mechanism of excitation of oscillations by gas scattering in weak-focusing synchrotrons by looking at the phase plane (x, x') where the phase trajectories are circles when the scale is properly chosen (fig. 55). The scattering corresponds to an instantaneous change of x' by constant x, that is a jump of the representative point from one phase trajectory to the other along a

vertical path, and a change of oscillation amplitude. These jumps being statistically independent, the distribution of particles in the plane x, x' spreads out. Some particles fall out of acceleration as they arrive at the boundaries of the stability region, which are represented schematically in fig. 55 by the two vertical lines.

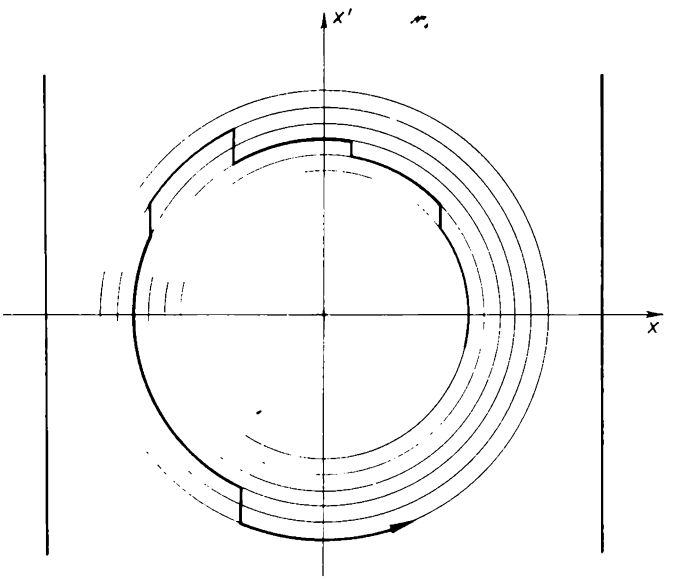

Fig. 55. Schematic diagram of multiple elastic scattering in the phase plane of betatron oscillations.

The proton elastic scattering cross-section is given by Rutherford's formula, according to which the mean square scattering angle is, for small angles, equal to

$$\overline{\chi^2} = \chi_{\min}^2 \lg (183 Z^{-\frac{1}{3}}) . \tag{2.9}$$

Here Z is the change of the scattering nucleus and $\chi_{\min}$ the minimum scattering angle due to screening by the electrons of the atom. The total elastic cross-section, also given by Rutherford's formula, is

$$\sigma = \frac{4\pi Z^2 e^4}{E^2 \beta^4 \chi_{\min}^2} ; \qquad \beta = v/c , \tag{2.10}$$

where v is the particle velocity. Thus, for proton elastic scattering, the quantity $\overline{\Delta y}$ becomes equal to

$$\overline{\Delta y} = \overline{\delta y} \cdot v \cdot \sigma N_0 = \frac{4\pi N_0 Z^2 e^4 c}{K_0^2 A_{\text{perm}}^2 E^2 \beta^3} \langle |f|^2 \rangle |f|^2_{\max} \lg (183 Z^{-\frac{1}{3}}), \quad (2.11)$$

N_0 being the number of atoms of the scatterer per unit volume*. We notice that $\overline{\Delta y} \approx (E^2 \beta^3)^{-1}$, so that scattering is especially large at low energies, that is in the non-relativistic region. Remembering that $dE_s/dt = (eV\omega_s/2\pi) \cos \varphi_s$ and assuming constant accelerating voltage and equilibrium phase, we obtain the following expression for τ for sufficiently large energies E:

$$\tau = \frac{8\pi^2 N_0 Z^2 e^4 \langle |f|^2 \rangle |f|^2_{\max} \lg (183 Z^{-\frac{1}{3}})}{K_0^3 A_{\text{perm}}^2 eV \cos \varphi_s} \int_{E_{\text{inj}}}^{E} \frac{dE}{E^2 \beta^4} \approx$$

$$\approx \frac{2\pi^2 N_0 Z^2 e^4 \langle |f|^2 \rangle |f|^2_{\max} \lg (183 Z^{-\frac{1}{3}})}{K_0^3 A_{\text{perm}}^2 eV \cos \varphi_s W_{\text{inj}}}, \quad (2.12)$$

where W_{inj} is the kinetic energy at injection (non-relativistic). Making use of this formula and of the graph in fig. 54 one may determine at once the degree of vacuum required for a given loss. In fact, if the residual gas is air at a temperature of 300 °K, then for $W \gg W_{\text{inj}}$ the pressure p, expressed in mm Hg, will be

$$p = 0.17 W_{\text{inj}} eV \cos \varphi_s \tau \frac{A_{\text{perm}}^2}{R^3 \langle |f|^2 \rangle |f|^2_{\max}}. \quad (2.13)$$

A_{perm} and R (mean radius of the machine) must be expressed in cm, and both W_{inj} and eV in kilo-electron-volts. For weak-focusing synchrotrons one may write $|f|^2 \approx |f|^2_{\max} = 1 \nu = 1/n$. With the typical parameters of large proton synchrophasotrons one finds, not taking into account damping, that the vacuum should be a few times 10^{-6} mm Hg for a permissible particle loss of 10%.

In the case where the losses are small, that is for small τ, one may estimate the effect of damping as follows. The losses being small, most

* It is assumed that the maximum scattering angle, determined by the nucleus dimensions, is smaller than the permissible angle for a single scattering (see § 5).

particles lie in the region of small u, far from the limit $u = 0$ where they get lost. So the distribution should be the same as if there were no upper limit. One can check immediately that the solution of eq. (1.5) with the boundary value zero at infinity and the initial condition $U(u, 0) = \delta(u)$ has the form

$$U(u, \tau) = \frac{1}{\rho^2(\tau)} e^{-u/\rho^2(\tau)}, \tag{2.14}$$

where

$$\rho^2 = D^2 \int_0^t \overline{\Delta y} \, dt = D^2 \int_0^\tau D^{-2} d\tau \tag{2.15}$$

is the mean square of the amplitude calculated with damping. In addition, the number of particles lost per unit time is proportional to the diffusion part of the flow through the boundary $u = 1$, that is

$$\frac{dN}{d\tau} = N \left(\frac{\partial U}{\partial u} \right)_{u=1} = - \frac{N}{\rho^4(\tau)} e^{-1/\rho^2(\tau)}, \tag{2.16}$$

Hence,

$$N(\tau) \approx \exp \left[- \int_0^\tau \rho^{-4}(\tau) e^{-\rho^{-2}(\tau)} d\tau \right]. \tag{2.17}$$

One can calculate the quantity $\rho^2(\tau)$ with (2.11), taking into account the practical form of the damping, which in the case considered is $D^2(t) = (W_{inj}/W)^{\frac{1}{2}}$, W being the non-relativistic kinetic energy. The result of the computation is also drawn in fig. 54 (curve 3). We observe that formula (2.17) gives a lower value of the losses, because the accuracy is lower the larger $\rho^2(\tau)$. The true curve must lie between curves 1 and 3 in fig. 54, approaching curve 3 for small τ.

Losses due to other processes involving the residual gas are calculated in the same way. Here one should keep in mind that for light particles (electrons) a large contribution to the losses may come from, besides elastic scattering, non-elastic collisions, accompanied by 'Bremsstrahlung', excitation of gas atoms, etc. Detailed calculations on this subject may be found in [8], [9].

3. THE CASE OF STRONG DAMPING. LOSS OF PARTICLES BY QUANTUM FLUCTUATIONS OF RADIATION

We shall now examine the other extreme case, where the damping is so strong that the mean square of the remaining amplitude is much less than the mean square of the permissible amplitude. For physical reasons it appears that most of the particles concentrate at small amplitudes and that the distribution soon becomes stationary.

The case of strong damping presents a particular interest for electron accelerators of high energies. As was shown in the previous chapter, a stationary amplitude is rapidly established under the simultaneous action of quantum fluctuations and radiation damping. It is natural to ask how large this stationary amplitude will be, with a view to estimating the particle losses which, of course, occur also in this case.

It is interesting that one may answer this question when $\kappa^2 \gg 1$ without finding directly the distribution function [10-12]. First of all we observe that the influence of the boundary where particles are lost at $u = 1$ must be weak, as the major part of the particles are concentrated in the region $u \lesssim \kappa^{-2}$. This indicates that the distribution must be nearly that calculated without taking the boundaries into account, in other words for diffusion into free space. As we saw in the preceding paragraph, such a distribution satisfying eq. (1.5) and zero boundary conditions at infinity is

$$U(u, \tau) = \rho^{-2} \, \mathrm{e}^{-u/\rho^2}, \tag{3.1}$$

or

$$U(u, \tau) \approx U_0(u, \tau) = \kappa^2 \, \mathrm{e}^{-\kappa^2 u}, \tag{3.1a}$$

where ρ^2 is the mean square value of the amplitude not taking the boundary where the particles are absorbed into account, and κ^{-2} its stationary value.

We notice that distribution (3.1a) remains for a time of the order of the damping time. As this time is short in the case considered one may neglect losses occurring during the time taken to establish stationary conditions. As quantity u is proportional to the square of the oscillation amplitude, that is to its energy, the distribution (3.1a) is not different from the Boltzmann distribution describing statistically the equilibrium position of a damped oscillator excited by stochastic processes. As the

absorbing boundary $u = 1$ lies at a large distance compared with the width of distribution (3.1a), once can expect that the exact distribution will differ very little from (3.1a).

After these introductory remarks, we find that according to the diffusion equation (1.5) the flux of particles at point u is equal to

$$I(u) = -u \frac{\partial U}{\partial u} - \kappa^2 u U , \tag{3.2}$$

and the number of particles lost per unit of time τ is

$$-\frac{dN}{d\tau} = NI(1) . \tag{3.3}$$

To find $I(1)$ let us multiply (3.2) by $e^{\kappa^2 u}/u$ and integrate between 0 and 1. We obtain

$$\int_0^1 \frac{I(u)}{u} e^{\kappa^2 u} du = U(0) . \tag{3.4}$$

Physically $I(u)/u$ can be considered as a function without singularity. As for the exponential under the integral, it increases rapidly with u, the more so the higher κ^2. Thus it is possible to take $I(u)/u$ outside the integral with its value at $u = 1$, so that we obtain for large κ^2

$$I(1) \approx \kappa^2 e^{-\kappa^2} U(0) \approx \kappa^2 e^{-\kappa^2} U_0(0) = \kappa^4 e^{-\kappa^2} . \tag{3.5}$$

Comparing this expression with (3.3) we find the number of particles remaining at the instant τ

$$N(\tau) = \exp\left[-\int_0^\tau \kappa^4 e^{-\kappa^2} d\tau \right] . \tag{3.6}$$

Let us now make use of the results obtained so far to calculate the loss of particles due to radiation quantum fluctuations. As we have found out already in ch. 5 § 5, losses due to the excitation of synchrotron oscillations are much more dangerous than losses of other types due to radiation fluctuations. So we shall examine only this kind of oscillation, for which the mean square amplitude value is given in eq. (5.27) or (5.29) of ch. 5, and the damping decrement in (3.14) of the same chapter. As

parameter κ^{-2} was defined as the square of the remaining amplitude, measured in units of the permissible amplitude, we find from (5.29) of ch. 5

$$\kappa^2 = \frac{32 r_e K_0 G_s \langle K^2 \rangle \gamma \eta_{\text{perm}}^2}{55\sqrt{3}\, \Lambda q \alpha \langle K^3 \rangle \operatorname{ctg} \varphi_s}. \tag{3.7}$$

The quantity η_{perm}^2 which is to be put into this formula depends on the practical conditions. In many cases the permissible amplitude of phase oscillations is determined by the dimensions of the stability region in phase space, i.e. by the bucket. We are then faced with a difficulty due to the fact that we have always assumed the oscillations to be linear whereas they are quite non-linear in the neighbourhood of the separatrix. One may overcome this difficulty by the following considerations. First, one can approximate the potential well of phase oscillations by the potential of the harmonic oscillator [13] and put

$$\eta_{\text{perm}}^2 = \frac{4(\sin \varphi_s - \varphi_s \cos \varphi_s)}{\sin \varphi_s}. \tag{3.8}$$

Alternatively, one may reason as follows. According to (3.5) the whole flux through the boundary $u = 1$ is equal to the 'diffusion' part of the flux through the same boundary calculated for the Boltzmann distribution

$$I(1) = -u \frac{\hat{c}}{\hat{c}u} U_0(u)\Big|_{u=1}. \tag{3.9}$$

At the same time, in the general case of non-linear oscillations, the stationary Boltzmann distribution is

$$U_0(\mathfrak{g}) = \frac{1}{\bar{\mathfrak{g}}} \exp\left[-\frac{\mathfrak{g}}{\bar{\mathfrak{g}}}\right], \tag{3.10}$$

where $\mathfrak{g}$ is the energy of the oscillatory process and $\bar{\mathfrak{g}}$ its mean value, which is established under the action of the stochastic perturbation. If $\mathfrak{g}$ is measured in terms of the maximum permissible energy $\mathfrak{g}_{\max}$, it is natural to replace u in (3.9) by $\mathfrak{g}$. This does not change anything for linear oscillations because the square of the amplitude u coincides with the energy $\mathfrak{g}$ in this case. At the same time the energy of the synchrotron

oscillations coincides with its Hamiltonian (ch. 4, eq. 7.16) and is, measured in units of $\mathfrak{g}_{max}$, equal to

$$\mathfrak{g} = \frac{(\varphi_0 - \varphi_s)\cos\varphi_s - (\sin\varphi_0 - \sin\varphi_s)}{2(\sin\varphi_s - \varphi_s\cos\varphi_s)}, \tag{3.11}$$

where φ_0 is the phase corresponding to the maximum excursion from the stable phase in the course of the synchrotron oscillation. With this in mind, one may consider that the quantity κ^2 in (3.5) and (3.6) is equal to 1 $\mathfrak{g}$ for non-linear oscillations or, as $\bar{\mathfrak{g}}$ corresponds to linear oscillations in the case of strong damping, to

$$\kappa^2 = \frac{1}{\bar{\mathfrak{g}}} \approx \frac{4(\sin\varphi_s - \varphi_s\cos\varphi_s)}{(\varphi_0 - \varphi_s)^2\sin\varphi_s} = \frac{\eta^2_{perm}}{(A^2_\eta)_{stat}}. \tag{3.12}$$

Thus, after the non-linearity of the oscillation has been taken into account in such an approximate manner, the quantity η^2_{perm} in (3.7) becomes equal to

$$\eta^2_{perm} \approx \frac{4(\sin\varphi_s - \varphi_s\cos\varphi_s)}{\sin\varphi_s}, \tag{3.13}$$

which gives the same result as (3.8), as can easily be proved. Introducing (3.13) in (3.7) we obtain finally

$$\kappa^2 = \frac{96eVG_s(\sin\varphi_s - \varphi_s\cos\varphi_s)K_0^2}{55\pi\sqrt{3}\,\hbar c\alpha q\langle K^3\rangle\gamma^3}. \tag{3.14}$$

This manner of taking the non-linearities into account is not strictly accurate, as one assumes implicitly that the non-linear oscillations are excited by the fluctuations in just the same way as the linear ones. However, for estimations in practical cases it is sufficiently accurate (see also [23]).

Depending on how conditions are in practice, there may be cases where the permissible phase oscillation amplitude is determined by other factors than the bucket dimensions, as for instance the displacement of the orbit from its equilibrium position. First, the orbit should not come near to the chamber walls, and secondly, the synchrotron oscillation should not be so large that betatron oscillations could approach dan-

gerous resonances. Such cases are, however, relatively rare, so we shall use (3.14) for further estimates.

The quantity $\cos \varphi_s$ entering the formulae is equal to

$$\cos \varphi_s = \frac{2\pi(W_s + \dot{E}_s)}{\omega_s e V}, \tag{3.15}$$

where the electron energy E_s and the loss power W_s are functions of time determined in a unique manner by the time dependence of the guide field value on the orbit. This dependence is given by the peculiarities of the magnet system of the synchrotron, in particular by field repetition rate.

As regards the quantity V – amplitude of the accelerating voltage – its time dependence is chosen so as to obtain the largest possible current and to use the power of the radiofrequency accelerating system with the maximum efficiency. Two typical extreme cases are to have V constant and φ_s constant during the acceleration cycle. In large synchrotrons it is evidently not rational to work with constant V, as this leads to too large an amplitude of radial synchrotron oscillations at injection, and requires unnecessary radiofrequency power. As the radiation loss rate increases rapidly with energy ($W_s \propto E^4$) it is necessary to vary V in the same manner $V \propto E^4$ if one wants to keep the equilibrium phase constant.

Figure 56 shows the mean beam lifetime for various radiation quantum fluctuations.

Figure 57 shows how the number of remaining particles depends on their energy for various values of the voltage amplitude V or of the equilibrium phase. As an example we have taken two synchrotrons: one operating, of 1.5 GeV (California, U.S.A., [12, 14]) and another, under construction, of 6 GeV (Cambridge, U.S.A., [11]). The latter machine will have magnetic biasing, with an energy variation with time as follows

$$E_s = E_{max} \sin^2 \pi f_B t, \tag{3.16}$$

f_B being the frequency of variation of the magnetic field and E_{max} the electron energy corresponding to the maximum guide field value. The graphs show that at high particle energies the loss rate can depend quite critically on the accelerating voltage.

To conclude this section let us note that when calculating the losses

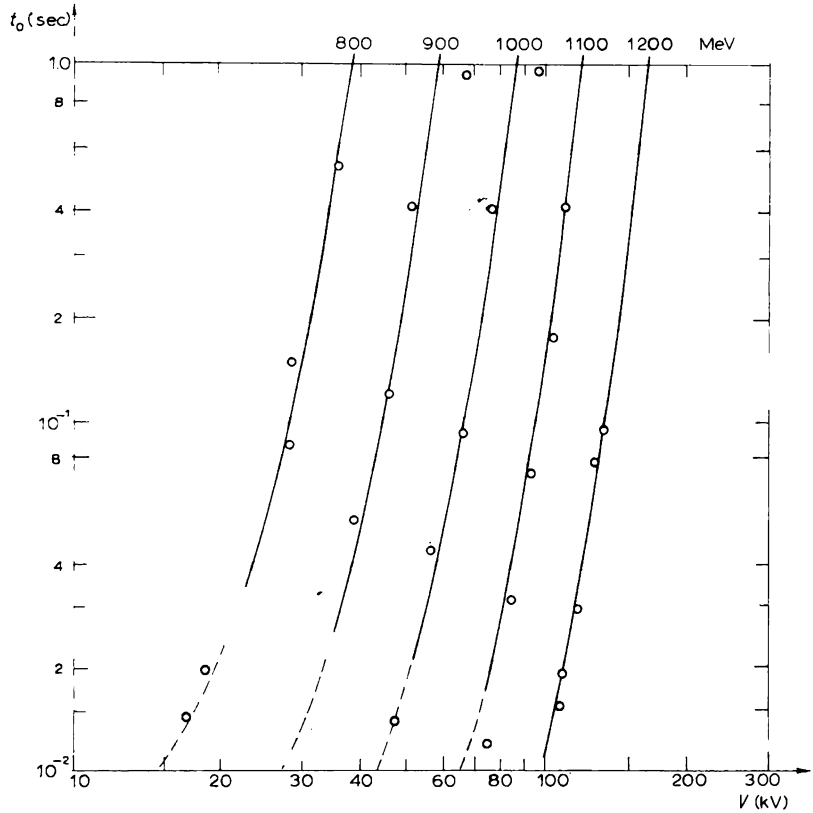

Fig. 56. Beam lifetime t_0, limited by quantum fluctuation losses, for an electron synchrotron of 1.5 GeV [14]: $q=4$, $\lambda=0.26$, $\alpha=2.5$, $2\pi/cK_0=0.96\times10^{-7}$ s, $\Delta W_{\text{turn}}=50$ keV $(E/1\text{ GeV})^4$.

due to radiation fluctuations one may with good reason completely neglect the losses corresponding to the emission of a single quantum, but not of many quanta, as is the case in the diffusion approximation which we are using here. This circumstance expresses itself in peculiarities of the radiation spectrum (see ch. 5, eqs. (1.6), (1.7) and fig. 45) by which losses caused by a single emission can be compared with diffusion losses only for very large quanta, whose probability is extremely small,

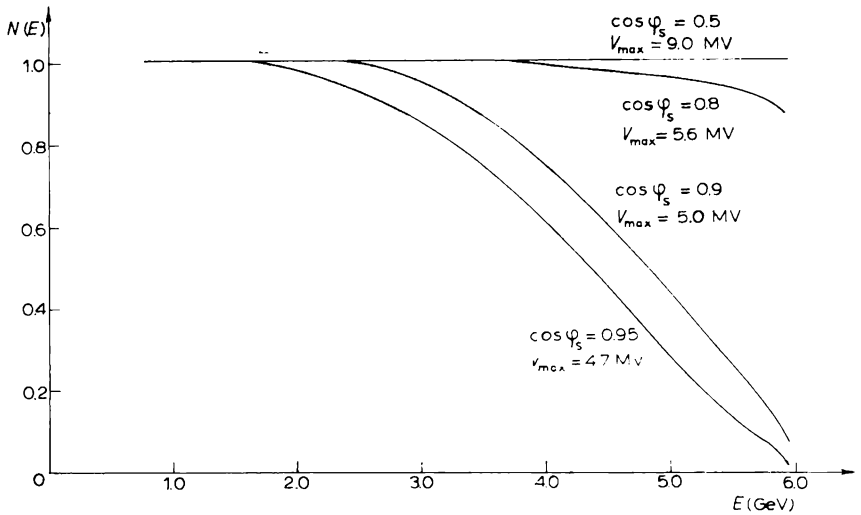

Fig. 57. Losses of particles by radiation quantum fluctuation in a 6 GeV electron synchrotron [11] for different equilibrium phases: $q = 360$, 1 $K = 25.85$ m, 1 $K_0 = 35.1$ m, $\alpha = 0.042$, $f_B = 60$ c/s.

as it decreases exponentially for $\omega > \omega_c$ (see ch. 5, eqs. (1.7), (1.9)). We have here an important difference from Coulomb scattering, where the probability decreases only with a power law, so that single scattering is often the most important part (see § 5).

4. GENERAL REMARKS ON THE INFLUENCE OF DAMPING ON THE LOSSES

When function $\kappa^2(\tau)$ is arbitrary, it is impossible to find a solution of eq. (1.5) in analytical form. One has then to apply numerical methods and take into account the practical features of the machine. However, in some cases one can make use of the fact that $\kappa^2(\tau)$ is a slowly varying function of τ in the region where most of the losses occur (actually κ^{-2} is the stationary value of the mean square of the amplitude and depends only on the instantaneous value of the machine parameters which vary slowly).

To begin with let us consider the losses when κ^2 is any given constant. This case has already a meaning in itself as it reflects the situation in storage rings. The particle energy is then practically constant, but the particles nevertheless experience damping because of the radiation effects.

Equation (1.5) lends itself, when κ^2 is constant, to separation of the variables, and has a solution in the form [15, 16, 22]

$$U = \mathrm{e}^{-\kappa^2 u} \Sigma C_i \mathrm{e}^{\kappa^2 \lambda_i \tau} F(\lambda_i, 1, \kappa^2 u), \tag{4.1}$$

where $F(\lambda, 1, \kappa^2 u)$ is the degenerated hypergeometric function [17] and the negative quantities λ_i are the eigenvalues of the problem, that is the roots of the equation

$$F(\lambda_i, 1, \kappa^2) = 0. \tag{4.2}$$

The number of particles remaining in the acceleration regime at the time τ is

$$N(\tau) = \int_0^1 U \,\mathrm{d}u = \Sigma C_i \mathrm{e}^{\kappa^2 \lambda_i \tau} \int_0^1 \mathrm{e}^{-\kappa^2 u} F(\lambda_i, 1, \kappa^2 u) \mathrm{d}u = \Sigma C_i \mathrm{e}^{\kappa^2 \lambda_i \tau - \kappa^2} F(1+\lambda_i, 1, \kappa^2). \tag{4.3}$$

The constants C_i are determined by the initial amplitude distribution of the particles

$$C_i = \frac{1}{N_i} \int_0^1 Y_0(u) F(\lambda_i, 1, \kappa^2 u) \mathrm{d}u, \tag{4.4}$$

where N_i is the norm of the hypergeometric function

$$N_i = \int_0^1 \mathrm{e}^{-\kappa^2 u} F^2(\lambda_i, 1, \kappa^2 u) \mathrm{d}u. \tag{4.5}$$

The dependence of the first eigenvalues of the problem (the smallest in modules) on κ^2 is shown in fig. 58. One sees that all λ_i are negative and increase rapidly in absolute value when i increases. Thus, when one has a sufficiently large $\tau \gg [\kappa^2 |\lambda_2(\kappa^2)|]^{-1}$ one may neglect in the sum (4.5) all terms with exception of the first, which corresponds to the smallest $|\lambda_i|$, so that

$$N(\tau) \approx C_1 \mathrm{e}^{-\kappa^2 + \kappa^2 \lambda_1 \tau} F(1+\lambda_1, 1, \kappa^2). \tag{4.6}$$

If one determines λ_1 with the graphs of fig. 58, one can find by this formula the quantity of the particles remaining after a sufficiently large time τ. One is, of course, specially interested in the case where the losses are small, that is in small $|\lambda_1|$. One can then use the approximation

$$\lambda_1 \approx -[\bar{E}i(\kappa^2) - \ln \kappa^2 - 0.577]^{-1}, \tag{4.7}$$

where $\bar{E}i(\kappa^2)$ is the modified exponential integral tabulated in [5]. Formula (4.7) gives an error of 18% for $\kappa^2 = 2$, 12% and 7% for $\kappa^2 = 3$ and 4, respectively. For small κ^2 one may take an empirical formula for λ_1 [15]

$$\lambda_1 \kappa^2 \approx -[1 + 0.45(1-\kappa^2)], \tag{4.8}$$

which has sufficient accuracy for $\kappa^2 < 1.5$.

It is interesting to see how formula (4.6) agrees with the asymptotic expression (3.6) for large κ^2. When $\kappa^2 \to \infty$ the eigenvalues λ_k tend towards the negative integers $1-k$, so that one can neglect in the expansion all terms with $i \neq 1$ as soon as $\tau > \kappa^{-2}$. Also, it is

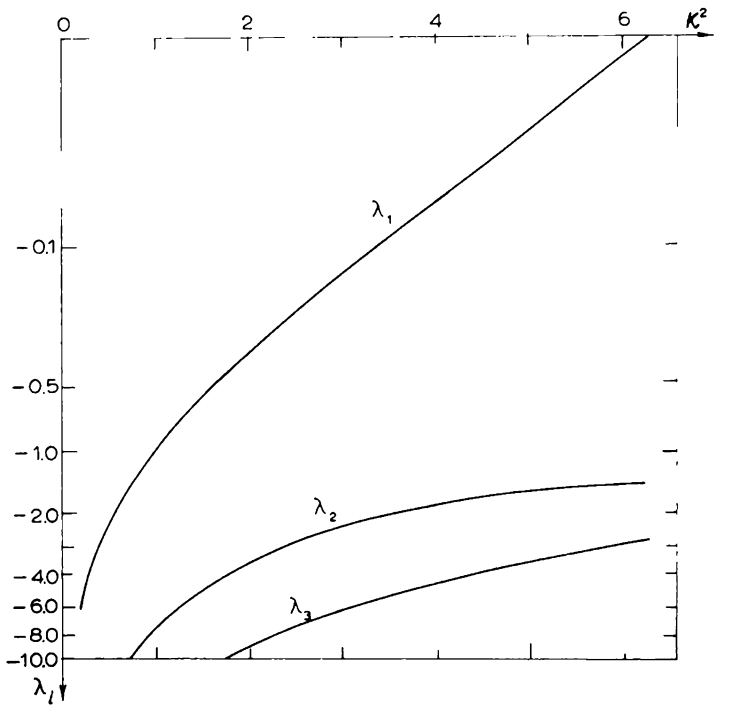

Fig. 58. Dependence of the first eigenvalues $\lambda_{1,2,3}$ on the parameter κ^2.

evident physically that the number of particles cannot decrease substantially in the course of a small interval τ, if their major part is concentrated at small u-values. Thus, for large κ^2 one can neglect all terms with $i \neq 1$ without large errors for any value of τ, and one can write

$$C_1 e^{-\kappa^2} F(1+\lambda, 1, \kappa^2) \approx 1 . \tag{4.9}$$

The physical meaning of this assumption is that during the very short interval $\tau \propto \kappa^2$ the distribution described by the function $e^{-\kappa^2 u} F(\lambda_1, 1, \kappa^2)$ is conserved, so that substantial losses have not occurred at the time when conditions become stationary. If λ_1 is small the distribution is nearly the Boltzmann distribution, as

$$e^{-\kappa^2 u} F(0, 1, \kappa^2) \equiv e^{-\kappa^2 u} . \tag{4.10}$$

With condition (4.9) we get

$$N(\tau) \approx \exp(\lambda_1 \kappa^2 \tau) , \tag{4.11}$$

or, using (4.7) and the asymptotic representation of $Ei(\kappa^2)$ for large κ^2

$$N(\tau) \approx \exp(-\kappa^4 e^{-\kappa^2 \tau}),$$

which is the same as formula (3.6) for $\kappa^2 = \text{const}$.

In order to obtain estimates in the more complicated case where κ^2 depends on τ, let us observe that the eigenfunctions of the problem $e^{-\kappa^2 u} F(\lambda_k, 1, \kappa^2 u)$ do not change very much when κ^2 varies, at least for large κ^2 when the λ_k are near negative integral numbers. If one neglects the variation, one can represent the solution in the form

$$U = \Sigma C_i \exp\left(\int_0^\xi \lambda_i \kappa^2 d\tau - \kappa^2 u\right) F(\lambda_i, 1, \kappa^2 u). \tag{4.12}$$

For this it is essential that the eigenfunctions vary very little in the range of the parameter τ, where the losses are important. This is the case, in particular, with losses due to radiation quantum fluctuations [16].

We obtain, in a similar way to that used above, a formula valid for sufficiently large τ

$$N(\tau) \approx C_1 \exp\left(\int_0^\tau \lambda_1 \kappa^2 d\tau - \kappa^2\right) F(1+\lambda_1, 1, \kappa^2), \tag{4.13}$$

and for large κ^2 we obtain formula (3.6), which is practically applicable for any value of τ,

$$N(\tau) = \exp\left[-\int_0^\tau \kappa^4 e^{-\kappa^2} d\tau\right].$$

It is of interest to investigate how weak damping affects particle losses. According to (2.7) the number of particles remaining after a sufficiently large time τ, is, in the absence of damping, proportional to

$$N(\tau) \propto e^{-\mu_1 \tau} = e^{-1.45\tau}, \tag{4.14}$$

where $\mu_1 = 1.45$ is the first root of eq. (2.2). At the same time, making use of (4.8) we find that, in the presence of weak damping which varies slowly,

$$A(\tau) \propto \exp\left(-\int_0^\tau (1.45 - 0.45\kappa^2) d\tau\right) = e^{-1.45\tau} D^{-0.9}(t). \tag{4.15}$$

To end this section we should like to emphasize again that more complicated cases than those reviewed above can crop up in practice. In particular, the parameter κ^2 can vary rapidly, or losses occurring simultaneously in several degrees of freedom, for instance scattering in the x- and z-directions, can be important [18]. The results obtained above can then only be used for initial estimates, and more exact calculations will require recourse to numerical methods.

5. LOSSES DUE TO SINGLE-SCATTERING EVENTS

In addition to small perturbations which make the oscillation amplitude increase slowly, an important contribution to the losses can come from events which send the particle almost instantaneously out of the working region. This applies in particular to scattering processes for which the large-angle scattering probability is relatively high, and falls at any rate much more slowly than the probability of emitting large quanta in losses by radiation fluctuation. It is thus interesting to dwell briefly on losses due to a single-scattering event.

Although the scattering cross-section is much larger for small angles than for large ones, single-scattering events can make an important contribution to the losses, as the particles are thrown out of the stability region at once, whereas small-angle scattering events lead to losses only if they are repeated many times. One should remember that small and large-angle scattering events always occur simultaneously and that it is extremely difficult to calculate the combined action of both. Strictly speaking, the results of the previous paragraph are valid only when the losses due to single scattering are negligible, or much smaller than the losses due to multiple scattering.

We shall now consider the other case, where the probability of losses by multiple scattering is deliberately assumed to be small, and we shall estimate the contribution from single scattering. In other words, we shall consider that the processes of single and of multiple scattering are independent and that the losses they cause are equal to the sum of the losses of both kinds, although this is true only if one of them is really predominant.

Such an approach is justified by the fact that if both kinds of losses are small, their sum will also be small, so that it is not very important to know their exact value. Op the other hand, the case where single- and multiple-scattering events are both large and comparable is not very interesting in practice, as it leads to very unfavourable conditions for the operation of an accelerator. One may find some considerations on the effect of single and multiple scattering occurring together in papers [8], [19], and [20].

It is not difficult to imagine practical conditions where single scattering would lead to much larger losses than multiple scattering. It is well

known that the scattering cross-section increases sharply according to a power law when one arrives at very small angles, and may be represented, when one takes correctly into account the screening effect of the atomic electrons, by the formula [7]

$$d\sigma = 2\pi \left(\frac{Ze^2}{E\beta^2}\right)^2 \frac{d\chi}{[\chi^2 + \chi_{min}^2]^{\frac{3}{2}}}. \tag{5.1}$$

for scattering in a plane. The minimum scattering angle χ_{min} is determined by the screening and is according to [21] equal to

$$\chi_{min} = \frac{3.7 \times 10^{-3} Z^{\frac{1}{3}}}{\beta E(\text{MeV})}. \tag{5.2}$$

We can use formula (5.1) also for single scattering, as the angles which would be considered as large from the point of view of accelerator operation are nevertheless small in absolute value because of the small aperture of the vacuum chamber. For the same reason one can neglect spin corrections, which are essential for large angles. More accurate expressions will be found in [8].

Single scattering will predominate if the minimum angle χ_{min} gives amplitudes comparable with the vacuum chamber dimensions, that is if

$$\beta E(\text{MeV}) < \frac{3.7 \times 10^{-3} Z^{\frac{1}{3}} |f|_{max}}{K_0 A_{perm}}. \tag{5.3}$$

For instance, in the 1.2 GeV strong-focusing synchrotron at Cornell, U.S.A., this inequality is fulfilled in the initial stage of acceleration, so that single scattering is important for the losses at that moment.

If $P(y,t)dt$ is the probability of having in the time interval dt a scattering event such that the amplitude of the betatron oscillation will exceed the permissible value, the equation for the distribution function can be written

$$\frac{\partial Y(y,t)}{\partial t} = -P(y,t)\, Y(y,t). \tag{5.4}$$

From this we obtain by a simple integration an expression for the number of particles remaining (without considering multiple scattering)

$$N(t) = \int_0^{y_{max}} \mathrm{d}y \cdot \exp\left[- \int_0^t P(y,t)\mathrm{d}t \right] Y_0(y) . \tag{5.5}$$

In order not to complicate the procedure, let us assume that all particles have zero amplitude at the initial instant. Then we find the simple expression

$$N(t) = \exp\left[- \int_0^t P(t)\mathrm{d}t \right], \tag{5.6}$$

where $P(t)\mathrm{d}t$ is the probability that in the interval $\mathrm{d}t$ the particle experiences scattering at a dangerous angle. If the particle had zero betatron oscillation amplitude up to the collision, the mean probable amplitude value after scattering at an angle χ will be

$$\overline{A} = \frac{1}{K_0} |f|_{max} \langle |f| \rangle \chi . \tag{5.7}$$

From this one easily finds the angle beyond which scattering will cause particle losses

$$\chi_0 = \frac{A_{perm} K_0}{|f_{max}| \langle |f| \rangle} . \tag{5.8}$$

It is possible to bring (5.7) into a more convenient form. Thus, if $\sigma(E, \chi)$ is the differential scattering cross-section for particles of energy E and N_0, the number of residual gas atoms per unit volume, we have

$$\int_0^t P(t)\mathrm{d}t = N_0 \int_0^t v \int_{\chi_0}^{\pi} \sigma(E,\chi)\mathrm{d}\chi\,\mathrm{d}t = \frac{2\pi N_0}{K_0 e V \cos\varphi_s} \times \int_{E_{inj}}^{E} \int_{\chi_0}^{\pi} \sigma(E,\chi)\mathrm{d}E\,\mathrm{d}\chi , \tag{5.9}$$

where it has been assumed that the quantities K, V and $\cos\varphi_s$ do not vary in practice in that stage of acceleration where most losses take place (at small energies).

As an example we shall give the results obtained for the Cornell electron synchrotron. This machine has an extremely small vacuum chamber height so that even for single scattering it is important to know the scattering cross-section at very small angles, for which the screening is essential. The experiment shows [9] that in the range of angles im-

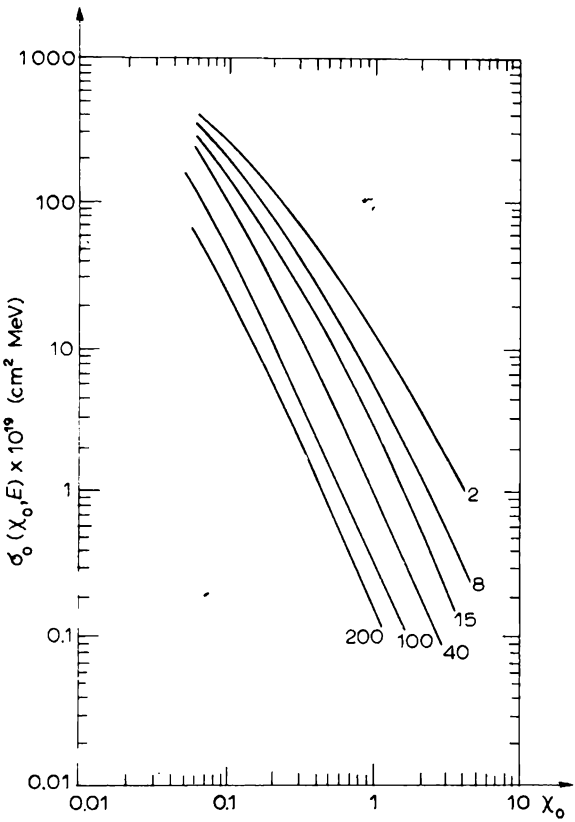

Fig. 59. Dependence of the total cross-section σ_0 on the angle χ_0. The numbers on each curve are the injection energies in MeV. Point $\chi_0 = 1$ corresponds to an angle of 5′ 43.8″.

portant in practice the inelastic scattering cross-section of the electrons can be equal to or even higher than the elastic scattering cross-section. The total cross-section, integrated over angle and energy, that is the quantity

$$\sigma_0 = \int_{E_{\mathrm{inj}}}^{E} \int_{\chi_0}^{\pi} \sigma(E, \chi) \mathrm{d}E \, \mathrm{d}\chi \,, \tag{5.10}$$

which enters into formulae (5.6) and (5.9) is represented in fig. 59 as a

function of the angle χ_0 and the injection energy E_{inj}. The integration was performed for an energy $E = 1$ GeV, but the result is of general interest, as the cross-section $\sigma(E, \chi)$ falls rapidly with increasing energy, so that the quantity σ_0 cannot depend strongly on the higher limit of integral (5.10). The figures obtained are given in table 5 for two limiting cases of initial distribution: zero initial amplitude (case A) and uniform distribution in betatron phase-space (case B). Multiple scattering does not contribute appreciably to the losses in this machine because of the relatively strong damping.

TABLE 5

Pressure in 10^{-6} mm Hg	Percentage of the particles lost by scattering			
	in radial direction		in vertical direction	
	A	B	A	B
10	31.0	29.5	51.8	70.8
5	17.0	18.0	30.6	46.0
2	7.2	6.8	13.6	21.8
1	3.7	3.5	7.0	11.6

The examples given in this chapter have necessarily each a special character, as in machines of different types different effects may cause particle losses and be present together. One can find in the literature quoted below more detailed investigations into this subject.

CHAPTER 7

PECULIARITIES OF ACCELERATORS OF DIFFERENT TYPES

A more or less thorough review of particle motion in accelerators has been made in the previous chapters. But attention was only drawn to the study of those properties of betatron and synchrotron oscillations which are inherent in all or most cyclic accelerators. In this chapter we shall study especially the features of the different types of accelerators which were not considered in detail, or were completely ignored in the general theory, in order not to disturb the flow of the discussion. Along with the application of the general formulae to concrete examples, and with some additional calculations in special cases, in this chapter specific methods of acceleration will be examined whose properties do not follow directly from the general theory (microtron, stochatron). However, we shall not treat the case of the electron synchrotron whose main feature is the essential role played by electromagnetic radiation, as this question has been dealt with in ch. 5 and partly in ch. 6.

1. ACCELERATORS OF THE CYCLOTRON TYPE

1.1. The cyclotron

The cyclotron was the first, and for many years the only type of resonance cyclic accelerator (see ch. 1 § 2). This is why its theory is well known and has been published in many review articles and books (see for instance [1], [2]). Here we shall restrict ourselves to the basic definitions of those characteristics of the cyclotron regime which differentiate it from other modern cyclic accelerators.

The concept of the cyclotron is very simple. An alternating potential

$$V_0 \cos \frac{2\pi}{T_0} t\,, \qquad T_0 = \text{const.} \tag{1.1}$$

is applied between two semicircular hollow electrodes, called dees (fig. 60). The dees are placed symmetrically in a vacuum chamber located between the poles of an electromagnet, which provides an azimuthally symmetrical magnetic field $\boldsymbol{B}$ constant in time. A particle passing through the gap between the dees receives an energy increment $eV_0 \cos \varphi$, where the φ is the phase of the electrical field at the time the particle passes

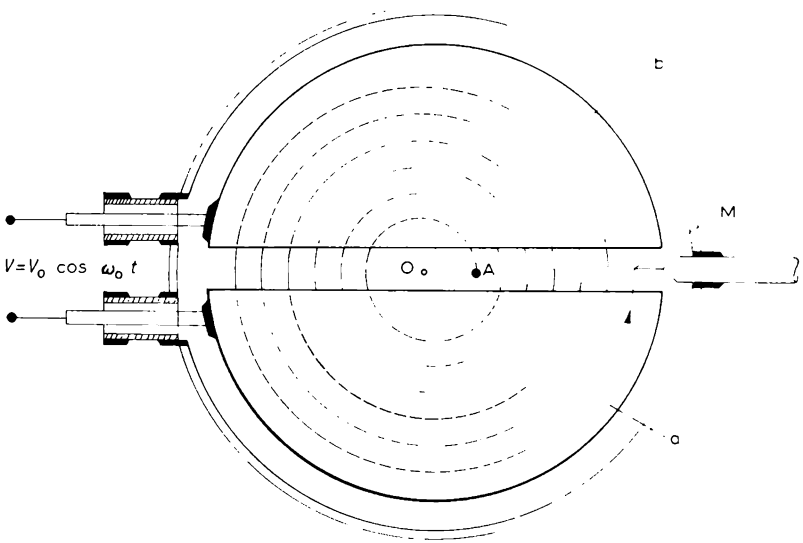

Fig. 60. Schematic representation of the spiral trajectory of an ion in the cyclotron. a, b, dees; A, ion source; M, target.

through the gap. The particle describes half a circle inside the dees under the influence of the field $\boldsymbol{B}$ and arrives at the diametrally opposite point of the gap where it experiences once more the action of the electric field.

The time $\frac{1}{2}T_0$ taken for the electrical voltage on the dees to change its polarity is made equal to the time of motion of the particle along the half-circle, i.e.

$$\tfrac{1}{2}T_0 \approx \tfrac{1}{2}T, \tag{1.2}$$

where T is the revolution period of the particle in the magnetic field. According to ch. 2 eq. (1.13), the expression for T may be written as

follows in the case of an azimuthally symmetrical magnetic field B

$$T = \frac{2\pi E}{ceB} \approx \frac{2\pi E_0}{ceB}, \tag{1.3}$$

where E is the total energy of the particle. For this reason, (1.2) presupposes the existence of a certain relationship between the parameters of the cyclotron:

$$B\lambda \approx 2\pi E_0/e, \tag{1.4}$$

where $\lambda = cT_0 = \text{const.}$ is the wave-length of the variable electric voltage. Under condition (1.4) the particle again receives an energy increase $eV_0 \cos\varphi$. travels along a half-circle of larger radius inside the second electrode, and so on. In this way, the particle moves outwards along a spiral of radius

$$r = \frac{E\beta}{eB}, \qquad \beta = \frac{v}{c}, \tag{1.5}$$

and gathers more and more energy in steps until it reaches the periphery of the accelerating electrodes or the edges of the pole-pieces of the electromagnet. The resonance acceleration regime starts from the centre of the magnet where the ion source is placed.

Let us examine first some features of the focusing of the particles in the cyclotron. The magnetic field in the central region is practically uniform (constant with radius) and therefore does not provide vertical focusing. In this region, that is, at small radii and energies, an important part will be played by the vertical focusing provided by the accelerating electrical field. In previous chapters we did not examine electric focusing, so we must study it now in more detail.

Let us consider the lines of force of the electric field $\mathscr{E}(s, z)$ between the dees at a given time t (see fig. 61), where s is the co-ordinate in the direction of flight of the particle in the neighbourhood of the point where it passes through the gap between the dees and z the vertical co-ordinate (the distance to the symmetry plane of the magnetic field). When moving in the plane $z = 0$, the particle experiences only the action of the component $\mathscr{E}_s$ of the field, which causes the particle energy to increase. However, when $z \neq 0$, the particle also experiences the action of the $\mathscr{E}_z$ component in the z-direction. As can be seen from fig. 61 a

particle to the left of the centre of the gap $s = 0$ is bent under the action $\mathscr{E}_z$ towards the median plane, whereas a particle to the right of the centre will be deflected away from this plane. The resulting deflection, however, is not zero because of two effects, which we shall call, rather arbitrarily, static and dynamic focusing.

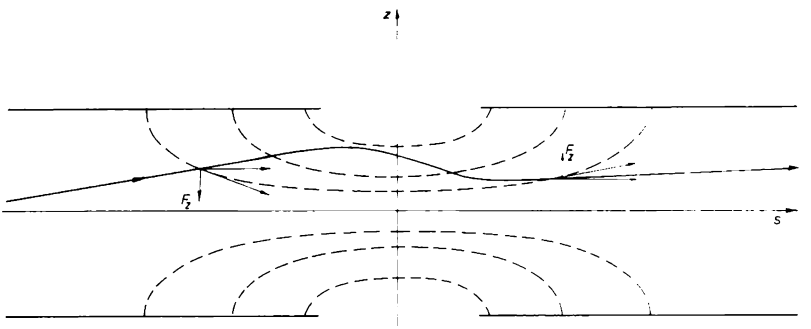

Fig. 61. Electric focusing and defocusing when the ion is crossing the gap between the dees. The dotted lines represent the lines of force of the electric field.

Static focusing is independent of the variation of the electric field with time. The effect of this kind of focusing, which is typical for electron optical lenses, is due to the change of the energy W and consequently of the velocity v of the particle at the time it crosses the gap. When the field is accelerating, the speed of the particle increases. This is why the particle will remain a longer time in the first region (focusing) up to the centre of the gap than in the second (defocusing). The overall result is a focusing effect.

The dynamic focusing is due to the change of the accelerating voltage with time. Thus, if the particle passes through the centre of the gap at a moment when the accelerating voltage is decreasing, it will travel during the first half of its path through a stronger electric field, and approach the median plane more than it will leave it in the second half.

Let us derive the formulae needed for quantitative estimates of the electrical focusing [3]. If one takes into account only the change in energy of the particle when it crosses the gap and not the variation of the electric field with time, one finds for the change $\Delta\chi$ in the slope of

the trajectory with respect to the median plane

$$(\Delta\chi)_{\text{stat}} \approx -C\left(\frac{eV_0}{W}\right)^2 \frac{z}{r}\cos^2\varphi\,. \tag{1.6}$$

Here the constant C is usually of the order of one, while W and φ are the kinetic energy and the phase of the particle when it crosses the centre of the gap.

If on the other hand one takes into account only the variation of the voltage with time, one finds

$$(\Delta\chi)_{\text{dyn}} \approx -\frac{eV_0 \sin\varphi}{Wr}\,z\,. \tag{1.7}$$

This formula shows that if $\varphi > 0$ (that is, when the voltage decreases with time) there is dynamic focusing, and if $\varphi < 0$ defocusing, whereby

$$(\Delta\chi)_{\text{dyn}} \gg (\Delta\chi)_{\text{stat}} \tag{1.8}$$

for all passages through a gap, except a few in the initial stage.

Thus, the gap between the dees acts as a lens on the motion of the particle in the z-direction. Let us now consider that the particle traverses a series of such gaps and assume that in the intervals in between it is not subject to vertical forces. We shall only retain $(\Delta\chi)_{\text{dyn}}$ and we shall take as independent variable the number k of gaps traversed. As the quantities χ and z vary rather slowly, we can make use of a differential equation instead of a difference one. The solution of the equation thus obtained

$$\frac{\mathrm{d}^2 z}{\mathrm{d}k^2} + \frac{\pi e V_0 \sin\varphi}{W}\,z = 0 \tag{1.9}$$

can be written as follows, because the parameters vary so slowly (see for instance ch. 2 § 6)

$$z = \text{const.}\left(\frac{W}{\sin\varphi}\right)^{\frac{1}{4}} \sin\left[\pi^{\frac{1}{2}} \int_0^k \left(\frac{eV_0 \sin\varphi}{W}\right)^{\frac{1}{2}} \mathrm{d}k + \delta\right]. \tag{1.10}$$

This expression shows that although there is electric focusing, the amplitude of vertical oscillations increases with energy as $(W/\sin\varphi)^{\frac{1}{4}}$, that is rather slowly, and also as $k^{\frac{1}{4}}$ when φ is constant. One sees also that in

the initial stage of acceleration in a cyclotron this kind of focusing plays an important part as, in its absence, z would grow linearly with W or k, that is much faster than with $k^{\frac{1}{2}}$. The frequency of the oscillation due to electric focusing is not large and, in agreement with (1.10), can be characterized by analogy with magnetic focusing by an 'electric' field index.

$$(\omega_z)_{el} = \omega\sqrt{n_{el}}\,, \qquad n_{el} = \frac{eV_0 \sin\varphi}{\pi W}\,. \tag{1.11}$$

As the particle energy and its distance from the centre of the magnet increases, the barrel shape of the magnetic field lines begins to be felt and the usual vertical magnetic focusing becomes effective (see for instance ch. 1 § 4). An equation which describes the simultaneous effect of electric and magnetic focusing can be expressed in the form

$$\frac{d^2z}{dk^2} + \pi^2[n_{el}(k) + n(k)]z = 0\,, \tag{1.12}$$

where the value of the magnetic-field index n increases gradually with radius although it remains a small quantity up to the final orbit ($n \ll 1$). Nevertheless, vertical magnetic focusing is of great importance in the cyclotron, as it limits the vertical beam dimensions, which increase at small radii where only electric focusing is effective.

A question which is vital in practice is the maximum energy of cyclotrons [3, 4] and with it the beam intensity. The acceleration regime in a cyclotron has the peculiarity which differentiates it from other types of resonance accelerators, that the equilibrium energy of the particles is a constant, because of the constancy of magnetic field and radio-frequency. We arrive at the same conclusion taking the general expression (3.17) of ch. 4, from which it follows that $\cos\varphi_s \equiv 0$ in a cyclotron and that in contrast with other accelerators the behaviour of the phase can be described by the pendulum equation without an external couple. Proceeding from this, it may be expected that the energy of the particle in a cyclotron would on the average be constant after a sufficiently long time, and then experience increases and decreases which can easily be identified with the synchrotron oscillations examined in ch. 4. So with a cyclotron one should aim finally to obtain particles whose energy should differ as much as possible from the equilibrium value in the course of such oscillations. Roughly speaking, the radial dimensions of

the bucket are comparable in this case with the dimensions of the working region, and the acceleration process lasts only half a synchrotron oscillation period. It is thus clear that the maximum energy W_{max} which can be reached by a particle must depend primarily on the amplitude of the accelerating voltage V_0.

Let us now consider cyclotron acceleration in more detail. Let the conditions for synchronism (1.2) and (1.4) be fulfilled at the centre of the magnet and the particles begin to be accelerated in resonance. As follows from (1.3), the revolution period increases with particle energy, an effect which is further enhanced by the fall of magnetic field with radius. The result is that the particle changes its phase continuously and finally falls into the region where the voltage becomes negative and begins to decelerate the particle. After the revolution period T has decreased enough to become again equal to the electric field period T_0, the particle phase begins to shift to the opposite side, that is to the left, and the procedure starts again in the opposite direction.

The maximum possible energy is, of course, reached if the particle with a variable phase spends as long as possible in the region where the voltage is accelerating. With this in view one should make the field B_0 at the centre of the magnet deliberately greater than the resonance field $B_{res}(0) = 2\pi E_0/\lambda e$ given by (1.4), i.e. $B_0 > B_{res}(0)$. It is then possible under certain conditions to have an acceleration where the particle starting with the phase $\varphi = \frac{1}{2}\pi$ and gradually moving to the left, moves down to $\varphi = -\frac{1}{2}\pi$ and turns back again. In doing this the particle passes twice over the region in which the voltage accelerates and gains a large amount of energy. However, under these conditions the beam intensity is vanishingly small as the initial phase region reduces to the point $\varphi = \frac{1}{2}\pi$. One can evidently increase the intensity by taking the initial phase region in the interval from $\varphi = \varphi_1 < \frac{1}{2}\pi$ to $\varphi = \frac{1}{2}\pi$ but the beam energy one can attain is somewhat reduced*. The definitive choice of the cyclotron parameters must be a compromise between the desire to reach a maximum particle energy and the need to get a sufficiently intense beam.

We shall now give an approximate calculation of the quantity W_{max}.

* Strictly speaking, one should consider the phase-focusing of the particles in the course of the first revolutions.

For simplicity we shall take a uniform field

$$B = B_0 = \text{const.}, \tag{1.13}$$

which is nearly true. We write the phase equation for the cyclotron in the following form, which can be derived immediately from (3.23) of ch. 4

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}k^2} - D\cos\varphi = 0, \quad D = \frac{\pi\omega_0 V_0}{cB_0}, \tag{1.14}$$

where k is as above the number of traversals of the accelerating gap. This equation is identical to the equation for the angular deviation of a pendulum, the equilibrium position of which corresponds to the angle $\varphi_s = \frac{1}{2}\pi$, counted from the horizontal plane through the point of suspension of the pendulum.

From (1.14) follows

$$\sin\varphi = \sin\varphi_i + H\frac{W}{eV_0} + \frac{D}{2}\left(\frac{W}{eV_0}\right)^2, \tag{1.15}$$

where φ_i is the value of the phase of the particles at the beginning of acceleration, that is at the centre of the magnet. $W = \int_0^k eV_0\cos\varphi\,\mathrm{d}k$ is the kinetic energy of the particles, and

$$H = \pi\left(\frac{B_{res}(0)}{B_0} - 1\right).$$

To obtain the highest energy W_{max} one should choose quantity H so that the phase φ varies from $\varphi_i = \frac{1}{2}\pi$ down to $-\frac{1}{2}\pi$, and then returns to its initial value in the course of acceleration. This requirement gives $H = -2D^{\frac{1}{2}}$ and leads to following expression for the maximum energy

$$W_{max} = 4\sqrt{(eV_0E_0/\pi)}. \tag{1.16}$$

As we see, the dependence of W_{max} on V_0 is not very sharp. Qualitatively this may be explained by the fact that although the increase in particle energy on passing through the accelerating gap is proportional to V_0, the number of gap traversals decreases as $V_0^{-\frac{1}{2}}$ when V_0 increases, as one can show. The resulting dependence of W_{max} on V_0 is as given above. The value of W_{max} for various kinds of particles is given in table 6. It

should be stressed that, in agreement with (1.16), these data correspond to an intensity dropping to zero. If one considers the decrease of the magnetic field towards the edge of the pole-pieces and requires that the intensity of the accelerated beam be large enough for experiments, the values given in table 6 must be divided by at least a factor of two.

TABLE 6
Theoretical maximum energy of a cyclotron

Particle	Rest energy (MeV)	W_{max}(MeV)	
		$V_0 = 100$ kV	$V_0 = 500$ kV
Protons	938	20	50
Deuterons	1876	30	70
Alpha-particles	3733	45	100

Thus the classical cyclotron, which has not lost its importance up to now, delivers ion beams of large intensity but limited to relatively low energies (of the order of a few MeV or of tens of MeV). Ways of increasing the maximum energy with cyclotron acceleration will be described in the next section.

1.2. The isochronous cyclotron

As was shown in the previous section the energy dependence of the particle revolution period in a cyclotron is such that substantial increase of the maximum energy requires extremely high voltages V_0. The use of these is however not practicable because of great technical difficulties.

Thus, it was extremely desirable to find a modification of the cyclotron, called isochronous cyclotron, in which the particle revolution period T does not depend at all on energy. In such a case the requirements laid down for V_0 no longer hold in principle, or at least cease to be determining.

From (1.2) to (1.5) it follows that an azimuthally symmetrical 'resonance' magnetic field B_{res}, which would ensure isochronism ($T = \text{const.}$), should increase with radius according to the law

$$B_{res}(r) = \frac{B_{res}(0)}{\sqrt{[1-(r/r_c)^2]}}, \tag{1.17}$$

where

$$B_{res}(0) = \frac{ce}{\omega_0 E_0}, \qquad r_c = \frac{c}{\omega_0}. \tag{1.18}$$

The magnetic-field index n must consequently depend on radius (and particle energy) in a determined manner

$$n = -\frac{r^2}{r_c^2 - r^2} = 1 - \frac{E^2}{E_0^2}. \tag{1.19}$$

At slightly relativistic energies one can consider that B_{res} should vary with radius approximately as

$$B_{res}(r) \approx B_{res}(0)[1 + \tfrac{1}{2}(r/r_c)^2 + \ldots]. \tag{1.20}$$

The radial dependence of a magnetic field of the type (1.17) or (1.20) is in direct contradiction with the requirement that an azimuthally symmetrical magnetic field should decrease with radius, as is necessary for vertical focusing. In the usual cyclotron it is thus impossible to maintain the isochronism conditions (1.2), (1.4) all the time and avoid the phase displacements of the particle which lead finally to formula (1.16) for the maximum energy W_{max}.

Methods of realizing isochronous cyclotrons have been known for a long time, but have been applied only in recent years. They consist of abandoning the azimuthal symmetry of the magnetic field, which must increase with radius on the average, and making use of azimuthal field variations so as to have vertical focusing of the particles. There are many different magnetic-field configurations which make it possible to construct such a cyclotron, the first of which was proposed by Thomas [5] as early as in 1938. He considered a field of the form

$$B(r, \theta) = B_0(1 + Ar \cos k\theta + Cr^2), \tag{1.21}$$

where A and C are suitably chosen constants and k is an integer. In such a field there exist stable orbits for energies up to a certain limit. This limit depends on the number k of field variations, which should in no case be smaller than three ($k \geqslant 3$). The modulation of the field with azimuth according to (1.21) brings about a periodic orbit excursion which in its turn causes a change of the restoring force of betatron oscillations in eq. (2.13) of ch. 2. The result is focusing with alternating sign which is in some ways analogous to strong focusing.

The means of producing the azimuthal field variation in isochronous cyclotrons are in principle similar to those applied to radial-sector and spiral annular strong-focusing accelerators (ring-phasotrons, see § 3). The main differences in the case of isochronous cyclotrons are first the absence of self-focusing (resulting from the concept of isochronism $\hat{c}T\ \hat{c}E \equiv 0$ itself) and second the breaking of the so-called scaling properties. It should also be mentioned here that the field increase with radius in such a cyclotron is usually much slower than in ring-phasotrons. One can divide the magnet systems used at present for isochronous cyclotrons into two categories: those with azimuthal variation of the field (with a form similar to eq. 1.21) and those with spiral variation of the field, in which the field varies periodically around a mean value not only with azimuth, but also with radius. This version can be more practical as it requires a smaller modulation of the magnetic field than in the case of the purely azimuthal variation.

So one has usually to express the magnetic field of an isochronous cyclotron in the following way

$$B(r, \theta) = B_0(r)[1 + \kappa U(r, \theta)], \tag{1.22}$$

where the quantity $B_0(r)$ must vary with radius according to the law which gives isochronism. The field variation required for the focusing is determined by the function $U(r, \theta)$ which is periodic in r and θ, has a mean value equal to zero and a maximum absolute value of the order of one. The parameter κ characterizes the field modulation ratio. For relatively small values of κ the function $B_0(r)$ is nearly equal to $B_{res}(r)$ from (1.17). For reasons of precision and simplicity of mounting it is desirable, although not necessary in principle, that the geometric law of the extreme values of the field $B(r, \theta)$ should be Archimedean (or linear) spirals $r = \text{const.} \times \theta$ (fig. 62). More complicated configurations for these lines, for instance logarithmic spirals (see § 3), are not necessary as orbit scaling is not maintained dynamically in the cyclotrons anyway.

To get some general figures, let us consider a field $B(r, \theta)$ modulated according to the following law

$$B(r, \theta) = B_0(r)[1 + \kappa \sin(N\theta - r/d)], \tag{1.23}$$

where $2\pi d$ is the increase of the radial co-ordinate of the spiral in one magnetic period. Introducing (1.23) into the equations for radial and

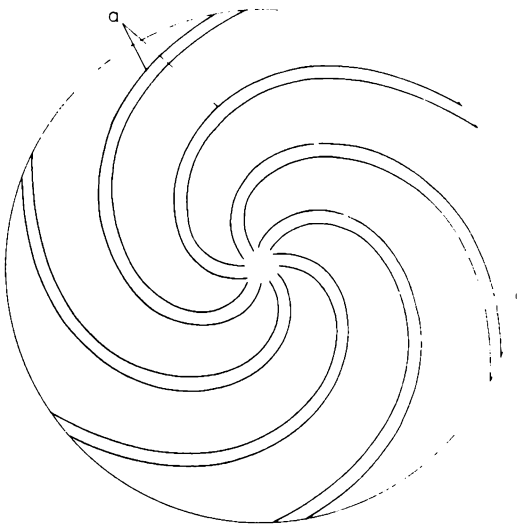

Fig. 62. Surface of the magnet pole-piece of an isochronous cyclotron (schematic). (a) the spiral shims which create the required magnetic-field configuration.

vertical motion one can by successive approximations find the closed-orbit equation and the linearized betatron oscillations with respect to this orbit [6, 7]. These equations can be written to a first approximation as follows:

$$x'' + \left[1-n-a + \frac{\kappa R}{d}\cos u(R,\theta)\right]x = 0\,, \tag{1.24a}$$

$$z'' + \left[n+a - \frac{\kappa R}{d}\cos u(R,\theta)\right]z = 0\,, \tag{1.24b}$$

where

$$a = \frac{\kappa^2 R^2}{2d^2[N^2-(1-n)]}\,, \qquad u(R,\theta) = N\theta - \frac{R}{d}\,, \tag{1.25}$$

and where R is the radius vector corresponding to the particle momentum and is given by the relation $p = (e/c)B_0(R)R$.

The equations (1.24) are related to Mathieu's equations, the funda-

mental properties of which were seen in ch. 2 §§ 3, 7. Assuming $\kappa R/N^2 d \ll 1$, $n \ll N^2$ we obtain for the betatron oscillation frequencies to a first approximation the relations

$$\nu_x^2 = 1 - n\,, \tag{1.26a}$$

$$\nu_z^2 = n + \tfrac{1}{2}\kappa^2 + (\kappa R/Nd)^2 = n + \tfrac{1}{2}\kappa^2 + \kappa^2 \operatorname{tg}^2 \delta\,, \tag{1.26b}$$

where δ is the angle between the spiral and the radius vector. Analogous expressions can be found in more general cases of spiral fields (see § 3).

The terms $1-n$ and n in the expressions for ν_x, ν_z come from the gradient of the average field $B_0(r)$. In the expression for ν_x the remaining terms compensate each other in practice; they come from edge-focusing at the ridges and effects due to the wavy form of the closed orbit. In the expression for ν_z three terms sum up and add themselves to the focusing term $\frac{1}{2}\kappa^2$, which remains even in the absence of spiral, that is in the case of radial sectors, where $\delta = 0$.

According to (1.19) and (1.26a) the quantity ν_x is directly related to the particle energy

$$\nu_x \approx E/E_0\,. \tag{1.27}$$

The motion of the particle begins with $E \approx E_0$ at the centre of the machine, where the frequencies of betatron oscillations are

$$\nu_x \approx 1\,, \qquad \nu_z \approx 0\,. \tag{1.28}$$

In the course of acceleration these frequencies increase. Apart from the central region of the magnet, the first non-linear resonances appear at the radius where $\nu_z \approx 0.5$ or $\nu_x \approx 1.5$. In particular, according to (1.27), radial oscillation resonances appear at energy values equal to an integral or half-integral multiple of the rest energy E_0. One finds also that the maximum particle energy one can reach is limited by the first main resonance encountered ($\nu_x = 2$), where the kinetic energy of the particle $W_{\max}$ is about equal to the rest energy $W_{\max} = E_0$.

If the accelerating voltage V_0 is large enough and the magnetic field correct, it may be that one can succeed in bringing the particles through resonances (see ch. 3 § 8) although this requires experimental confirmation. At any rate to remain within the stability region, one must respect the inequality $\nu_x < \frac{1}{2}N$ ($\mu_x < \pi$). The energy attainable cannot exceed in

the best case the value $\frac{1}{2}NE_0$ where N is the number of periods. This figure can be reduced to $\frac{1}{3}NE_0$ ($\mu = \frac{2}{3}\pi$) because of the non-linear cubic resonance (ch. 3 §§ 5, 7). In the central region of the magnet one can also have non-linear resonances of the type $k_x \nu_x = k_N N$, where $k_x = N$, $k_N = 1$, $\nu_x = 1$ in our case (see [7]).

It should be clear that the main parameters of the cyclotron, such as N, κ, δ, cannot be chosen independently and without considering convenience and simplicity of construction. As an example, one can only achieve a sufficient modulation κ, especially at small radii, if N is not large. In machines existing or being designed N is 3 to 4 for the azimuthal and 4 to 6 for the spiral field variation.

As regards the phase motion in an isochronous cyclotron, the very concept of isochronism excludes phase stability (self-focusing) so that $\mathcal{H} \equiv 0$ (see ch. 4, eq. 3.3). Thus, tolerances regarding magnetic-field perturbations and constancy of acceleration frequency over a large number of turns must be extremely severe and will result in technical difficulties in the construction of an isochronous cyclotron.

To make some estimates, let us assume that the law (1.17) is followed with some constant relative error $\Delta B/B$. Let us then require that the phase displacement of the particle caused by this error should not exceed $\frac{1}{2}\pi - \varphi_i$, so that the particle does not come into the region where it is decelerated.

Under these conditions the tolerance on $\Delta B\, B$ may be written in the form of inequality

$$\frac{\Delta B}{B} < \frac{eV}{2\pi E_0} \frac{\sin\varphi - \sin\varphi_i}{1-\sqrt{(1-\beta^2)}}, \tag{1.29}$$

where the initial phase value φ_i lies within the limits $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ corresponding to acceleration. The tolerance on the constant relative discrepancy between radiofrequency ω_0 and cyclotron frequency ω can be written as follows

$$\frac{\Delta\omega_0}{\omega} = \frac{eV_0}{2\pi W}(\sin\varphi - \sin\varphi_i). \tag{1.30}$$

One should keep in mind that the isochronism condition refers to the revolution period of the particle along the closed orbit, which differs more from a circle the higher the modulation coefficient κ and the lower

the period number N. Then, if one takes as 'mean' field $B_0(r)$ the field $B_{res}(r)$ according to (1.17), the phase displacement will be given, not only by inaccuracies in the shimming and the stabilization, but also by the systematic deviation of the closed orbit from a circle. For large values of k it is essential to introduce a correction into the law $B_{res}(r)$ in order to take this into account.

There exist variants other than the magnet systems we have considered which can be applied in principle to isochronous cyclotrons. Let us mention for instance the so-called sector-type magnet which consists of separate sectors in which the magnetic field is nearly uniform, as it is also in the intervals between them where it has another value [9]. The isochronism condition is reached in this case by giving a special form to the edges of the magnet sectors. Vertical focusing, which has in principle the same origin as with the azimuthal and spiral variations of the magnetic field, is obtained by boundary effects at the sector limits.

1.3. The phasotron

In principle, the phasotron (synchrocyclotron) differs from the cyclotron by the fact that the frequency ω_0 of the electric field varies (decreases) in the course of acceleration. Because of the self-focusing, this permits

TABLE 7

Basic parameters of the phasotron of the JINS

Maximum proton energy	680 MeV
Diameter of the pole-piece	6 m
Magnet gap (at the centre)	600 mm
Magnetic field (at the centre)	16.6 kgauss
External dimensions of magnet	18×10 m^2
Magnet weight	7000 t
Magnet supply power	1000 kW
Magnet current stability	0.1 %
Magnetic field decreases at the edge by	4.9 %
Amplitude of accelerating voltage	15 kV
Frequency range	26.5–13.6 Mc/s
Repetition rate of the accelerating cycle	100 c/s
Operating pressure in the tank	$6–7 \times 10^{-6}$ mm Hg
Mean current of accelerated particles	0.3 μA

a considerable increase of the energy of the accelerated ions compared with the maximum energy of the cyclotron (see eq. 1.16). The maximum energy in the phasotron, which can in some cases reach 600–700 MeV for protons [10], is limited mainly by the dimensions of the magnet. Also, since ω_0 varies with time, the capture of the ions into the acceleration mechanism occurs only in the course of short periods of time, separated by relatively long intervals equal to the modulation period of ω_0. Because of the pulsed character of the operation the beam intensity in the phasotron is substantially lower than in the cyclotron. The average particle current at the end of acceleration in the phasotron is of the order of 0.1–1 μA whereas it is 0.1–1 mA in the cyclotron (see table 7).

In phasotrons as in cyclotrons one has a compact (not an annular) magnet with an azimuthally symmetrical field $B(r)$. This field is not far from being uniform radially, but its decrease at the edge of the pole-pieces can be made much larger than in the cyclotron because of self-focusing (from 3 to 5%). This gives the possibility of providing sufficiently good vertical magnetic focusing for the particles.

The variation of the radiofrequency ω_0 with time must take place slowly enough to follow the change of the particle revolution frequency ω as the energy increases. More exactly, the quantity $\mathrm{d}\omega_0\,\mathrm{d}t$, according to (3.17) of ch. 4, has to satisfy the inequality

$$|\cos\varphi_s| = \left|\frac{\pi E_s}{qeV_0\mathscr{K}\omega_0^2(t)}\frac{\mathrm{d}\omega_0(t)}{\mathrm{d}t}\right| < 1\,, \tag{1.31}$$

which is the condition for the existence of an equilibrium phase and of self-focusing (usually the harmonic number q is equal to 1 in the phasotron, just as in the cyclotron). The quantity $\mathscr{K}$ in (1.31) is a known quantity, depending on field shape and particle energy, and equal, for azimuthal symmetry, to

$$\mathscr{K} = 1 + \frac{n}{1-n}\frac{1}{\beta^2}\,. \tag{1.32}$$

The equilibrium phase φ_s does not usually remain constant in the course of acceleration, as this would require special measures. For instance, for the simplest case of a uniform magnetic field $B = B_0 = \text{const.}$, the fre-

quency $\omega_0(t)$ would have to vary according to the following law, if φ_s were to remain constant

$$\omega_0 = \frac{\omega_{0i}}{\sqrt{\left[1 - \frac{2}{\omega_{0i}}\left(\frac{d\omega_0}{dt}\right)_i t\right]}}, \tag{1.33}$$

where ω_{0i}, $(d\omega_0/dt)_i$ are the initial values of the quantities considered. The requirement of a constant φ_s is not essential: it is only necessary to take care that φ_s varies slowly compared with the period of synchrotron oscillations.

The characteristic features of the phase motion and the synchrotron oscillations in the phasotron can be found with the help of the general theory developed in ch. 4. However, a few peculiarities of the particle motion, related for instance to the capture into the acceleration mechanism, must be examined separately.

Let us first consider the motion of the ion during the first turns [11] while its trajectory lies inside the accelerating gap ($r < b$, b being the gap width). During this time it is possible to consider the fields B and $\mathscr{E}$ as uniform and to neglect the variation of ω_0, so that one can write the equations of motion in the form

$$m_0\ddot{s} = \frac{eB_0}{c}\dot{y} - e\mathscr{E}_0 \cos(\omega_0 t + \varphi_i), \tag{1.34a}$$

$$m_0\ddot{y} = -\frac{eB_0}{c}\dot{s}, \tag{1.34b}$$

where the symbols are the same as in § 1.1. Evidently these equations will also describe the motion for $r < b$ in the cyclotron. The solution of eqs. (1.34) with the initial conditions $t = 0$, $s = \dot{s} = y = \dot{y} = 0$, $\varphi = \varphi_i$ is

$$s = -\frac{e\mathscr{E}_0}{m_0\omega\Delta\omega}\{\sin[(\omega + \tfrac{1}{2}\Delta\omega)t + \varphi_i]\sin\tfrac{1}{2}\Delta\omega t - \tfrac{1}{2}(\Delta\omega/\omega)\sin\omega t\sin\varphi_i\}, \tag{1.35a}$$

$$y = -\omega\int s\,dt, \tag{1.35b}$$

where

$$\omega = ceB_0/E_0, \qquad |\Delta\omega| = |\omega_0 - \omega| \ll \omega_0. \tag{1.36}$$

The first term in the expression for s is after 2 or 3 revolutions already much larger (about ωt times) than the second one. Thus, the times t_k at which the particle passes through the centre of the accelerating gap are given by the relation

$$(\omega + \tfrac{1}{2}\Delta\omega)t_k + \varphi_i \approx \pi k \,. \tag{1.37}$$

The phase of the electric field at the instant t_k, given by definition by $\varphi_k = \varphi_i + \omega_0 t_k - \pi k$ may be, according to (1.36) and (1.37), expressed in the form

$$\varphi_k \approx \tfrac{1}{2}\Delta\omega\, t_k \,. \tag{1.38}$$

If the frequency deviation is small ($\Delta\omega \approx 0$) as is the case in practice, one has $\varphi_k \approx 0$ during the first few revolutions. This shows that after 2 or 3 turns the particle phase becomes nearly equal to zero independently of the initial phase φ_i. Thus, a peculiar phase-focusing effect occurs, which pulls all the particles towards the phase $\varphi = 0$ corresponding to the maximum of the electric field.

We shall now study the capture of the particles into the acceleration mechanism [12, 13]. For this one must know the behaviour of the phase near the centre of the magnet, that is for radii which are small but larger than the width of the accelerating gap. As we are interested only in the motion during the first phase oscillation, one may consider that

$$E_s \approx E_0 = \text{const.}\,, \qquad \omega \approx \omega_c = eB_0 c/E_0 \,. \tag{1.39}$$

For small radii the shape of the magnetic field can be approximated well by a parabola

$$B(r) \approx B_0(1 - \tfrac{1}{2}hr^2)\,, \qquad \tfrac{1}{2}hr^2 \ll 1 \,, \tag{1.40}$$

where $h = \text{const.}$ If this is the case

$$n \approx hr^2 \,, \qquad \mathscr{K} \approx 1 + hc^2/\omega_c^2 \tag{1.41}$$

so that one may consider the quantity $\mathscr{K}$ to be constant also.

With the assumptions we have made, the phase equation can be written for small radii, according to ch. 4 § 3,

$$\frac{\pi E_0}{\omega_c^2 + hc^2}\,\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} - eV_0 \cos\varphi = -eV_0 \cos\varphi_s \,, \tag{1.42}$$

where

$$\cos \varphi_s = -\frac{\pi}{eV_0} \frac{E_0}{\omega_c^2 + hc^2} \frac{d\omega_0}{dt}. \tag{1.43}$$

Let us assume, in accordance with the conclusion reached above, that the particles which have started to move inside the accelerating gap at different times all have, after a few turns, the same phase $\varphi_i = 0$. As for the derivative $\dot{\varphi}_i$, it varies with time

$$\dot{\varphi}_i(t) = \omega_0(t) - \omega_c . \tag{1.44}$$

To be captured into the acceleration mechanism the particles must have initial phases φ_i and phase derivatives $\dot{\varphi}_i$ which lie within the region of stability given by the bucket. However, this is not sufficient. One must still ensure that in the course of its first phase oscillation the particle does not return to the centre of the magnet, as the probability of its being lost increases very much at small radii, through processes such as hitting the source, scattering or neutralization (change of charge) by residual gas atoms, etc. If the ion does not return to the centre after the first phase oscillation it will never do so since after each new oscillation its radius will be larger than it was before.

If the particle is to be inside the stable phase region, $\dot{\varphi}_i$ must lie between two limits: $-\dot{\varphi}_{max} < \dot{\varphi}_i < \dot{\varphi}_{max}$ where

$$\varphi_{max} = \sqrt{\frac{2eV_0}{\pi E_0} (\omega_c^2 + hc^2) F(\varphi_s)}, \tag{1.45}$$

$$F(\varphi_s) = \sin \varphi_s - \varphi_s \cos \varphi_s . \tag{1.46}$$

The expressions (1.45) and (1.46) were obtained from the first integral of eq. (1.42) for $\varphi_i = 0$. Let us note that with the help of (1.43) the expression for $\dot{\varphi}_{max}$ can be rewritten in the simpler form

$$\dot{\varphi}_{max} = \sqrt{2 \left| \frac{d\omega_0}{dt} \right| \frac{F(\varphi_s)}{|\cos \varphi_s|}}.$$

However, it is more convenient to study $\dot{\varphi}_{max}$ in its form (1.45), as $\cos \varphi_s$ is determined by the velocity of modulation $d\omega_0/dt$ when the other parameters remain the same. A graph of the function $F(\varphi_s)$ is

presented in fig. 63. It gives the range of permitted $\dot{\varphi}$, which is largest for $\varphi_s = \frac{1}{2}\pi$ and goes to zero for $\varphi_s = 0$.

On the other hand, when φ_s is sufficiently large there appears a range of $\dot{\varphi}_i$-values for which the particles are bound to return to the centre (i.e. to the source) and this range grows with φ_s. For small φ_s, initial

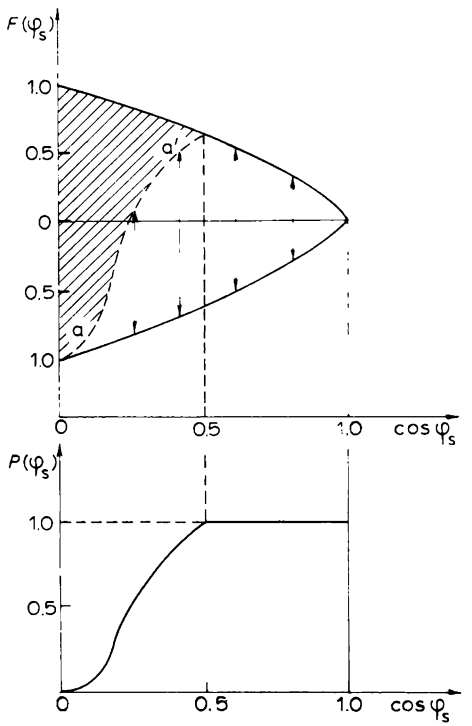

Fig. 63. Illustrating the calculation of the coefficient of capture into the synchrocyclotron regime.

conditions for which the particle is decelerated do not exist at all within the limits of the bucket. This is the case as long as φ_s is less than 43°. For large φ_s there exist already some values of $\dot{\varphi}_i$ for which the particle falls into the decelerating field and for $\varphi_s \gtrsim 60°$ some values of φ_i give total deceleration, that is return to the source. In fig. 63 the curve aa′ indicates

the limit $\dot{\varphi}_i = \dot{\varphi}_{lim}$ for which the particle trajectories arrive at the centre at the instant when the particle leaves the region of phase for which there is deceleration. In fig. 63 is also represented the function $P(\varphi_s)$ which is equal to the phase interval which permits acceleration without return to the centre divided by the whole interval $2\dot{\varphi}_{max}$ for a given φ_s

$$P(\varphi_s) = \frac{\dot{\varphi}_{max} + \dot{\varphi}_{lim}}{2\dot{\varphi}_{max}}. \tag{1.48}$$

Evidently the time during which capture can take place without the particle going back to the centre is, according to (1.45), equal to

$$\tau = \frac{2\dot{\varphi}_{max} P(\varphi_s)}{|d\omega_0/dt|} = \frac{2}{|d\omega_0/dt|} \sqrt{\frac{2eV_0(\omega_c^2 + hc^2)}{\pi E_0}}\, L(\varphi_s), \tag{1.49}$$

where

$$L(\varphi_s) = F^{\frac{1}{2}}(\varphi_s) P(\varphi_s). \tag{1.50}$$

Knowing τ and the cycle repetition rate f_{rep}, one can find the ratio of the particle current I which is accelerated, to the source current in the centre of the magnet

$$I/I_0 = \tau f_{rep}. \tag{1.51}$$

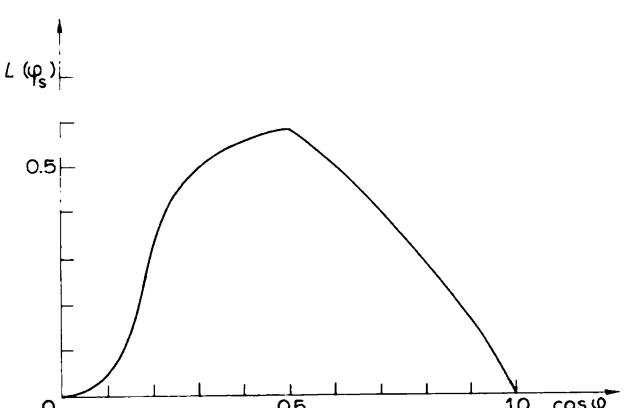

Fig. 64. Graph of the function $L(\varphi_s)$.

The ratio f_{rep} to $|d\omega_0/dt|$ is a constant for a given accelerator. So, as appears from (1.49), the largest value of I/I_0 for a given V_0 is obtained when φ_s is about to 60°, the value for which the function $L(\varphi_s)$ has its maximum. The complete graph of this function is shown in fig. 64. With the help of this graph and of formula (1.50) one can determine the optimum conditions of capture into the synchrocyclotron regime depending on parameters such as frequency modulation rate $d\omega_0/dt$, amplitude of the voltage V_0 and so forth. Experimental results agree with the figures derived from these theoretical considerations.

Several experiments with the phasotron gave the unexpected result [14] that a noticeable fraction of the particles was accelerated even when the inequality $\cos\varphi_s > 1$ and the normal conditions for resonance acceleration were not fulfilled. It was established that in this case the acceleration did not take place during one cycle of the modulation, but in the course of several. Particles which had once fallen out of synchronism in a given cycle and remained on the orbit of appropriate radius, were picked up again with a certain probability at one of the following cycles, and could so reach the maximum energy. Such a process is similar in character and as regards beam intensity to the stochastic regime which will be considered in § 4.

We should mention that when accelerating ions in the synchrocyclotron and when extracting them from the machine, in practice one runs into dangerous resonance phenomena in the region where the magnetic-field index n takes the value 0.2. These phenomena, which can lead in a number of cases to complete loss of beam, are of importance for the operation of the machine and have been well studied experimentally (see [15]).

When n is equal to 0.2 one has

$$\nu_x - 2\nu_z = \sqrt{1-n} - 2\sqrt{n} = 0\,, \tag{1.52}$$

corresponding to a cubic difference resonance, which does not depend on field perturbations, that is on deviations from symmetry (see ch. 3 §§ 4 and 7. The coupling of the x- and z-oscillations is determined by the parameter

$$n_1 = \frac{R^2}{B}\left(\frac{d^2B}{dr^2}\right)_{r=R} = \left(n^2 + n - r\frac{dn}{dr}\right)_{r=R}, \tag{1.53}$$

where R is the radius on which (1.52) is fulfilled. This parameter, which characterizes the variation of the magnetic-field gradient with radius, enters into the betatron oscillation equation as follows:

$$\text{(a)}\quad x''+(1-n)x=\tfrac{1}{2}n_1 z^2/R\,,\qquad \text{(b)}\quad z''+nz=n_1 xz/R\,. \tag{1.54}$$

This resonance, as well as the other difference resonances, consists of a transfer of energy from the x- to the z-oscillation and vice versa. This phenomenon can be dangerous as the space available for the z-oscillation (gap between the dee plates) is much smaller than the space available in the radial direction. Thus a transfer of the energy of the radial motion into vertical motion can be accompanied by large particle losses. The coupling parameter n_1 determines the beat period of the amplitudes A_x and A_z

$$T_{\text{beat}} \approx T\,\frac{\sqrt{1-n}\,R}{|n_1|\,A_x}\,, \tag{1.55}$$

which can be found from (1.54), for instance, by averaging (T is the period of revolution).

When estimating the effect of the resonance (1.52) one should keep in mind that the orbit radius of the ion increases with energy. In the course of this the quantity $n=n(r)$ and accordingly ν_x and ν_z vary and the ion which has entered the resonance region moves gradually out of it. In order to characterize this process, which was considered in ch. 3 § 8, one can introduce a period T_{res} equal to the time during which the resonance takes place, that is during which a noticeable change of the amplitudes A_x and A_z occurs. We find for T_{res} the expression

$$T_{\text{res}} \approx 4\pi\sqrt{2n}(1-n)\,T\,\sqrt{\frac{W}{R(\mathrm{d}n/\mathrm{d}r)_R\,\Delta W}}\,, \tag{1.56}$$

where ΔW is the energy gain per turn and W the particle energy.

If $T_{\text{beat}} < T_{\text{res}}$ there is exchange of energy between the x- and z-oscillations in the time needed to pass through resonance. On the other hand the resonance at $n=0.2$ will be practically without danger. For instance, in the article considered [15] it is mentioned that for the 184 inch Berkeley cyclotron where $n_1=-6.4$, T_{beat} is about $10T$, whereas T_{res} is

equal to $100T$, so that the beam intensity falls rapidly when the region where $n = 0.2$ is reached. Of course, particles with sufficiently small initial radial amplitudes will be able to pass through this region. However, in this machine these particles will be lost in the vertical direction due to the effect of the parametric resonance ($\nu_3 = \frac{1}{2}$) for $n = 0.25$ in a region which comes immediately after the region where $n = 0.2$.

To conclude this section, let us say a few words about the extraction of the beam from the machine. In a cyclotron one makes use of an electrostatic deflector (condenser) which is placed tangentially to the last ion orbit. Because of the relatively large radial pitch of the spiral trajectory the ions are able to pass behind the deflector plate, which has a finite thickness. Travelling further under the action of both electrostatic and magnetic fields, the ions come out of the machine. In principle the same method can be used for extraction from the synchrocyclotron. However, it runs into serious difficulties in practice. This is due to the fact that the spacing of the orbits in the synchrocyclotron is much smaller than in the cyclotron because of the smaller accelerating voltage V_0 and the higher energy W of the ions.

So one is obliged to use more complicated methods of extraction, aimed at increasing orbit spacing at the end of acceleration. The most efficient one is 'regenerative' extraction [16, 17]. The idea is to create at the appropriate radius a regime of particle motion where the radial oscillations become unstable, whereas the vertical ones remain stable as before. This is usually obtained by introducing two local perturbations of the magnetic-field index placed at different azimuths on the last orbits. At one azimuth one creates a relatively large positive field gradient, and at the other a negative one.

With a good choice of the parameters involved, the radial orbit excursions will increase exponentially and, because of the large orbit spacing, after a few turns the particles will be able to pass into the screened magnetic channel and come out.

1.4. An accelerator with variable harmonic number: the microtron

In the cyclic resonance accelerator with variable harmonic number (microtron), just as in the cyclotron, the magnetic field B and the period T_0 of the accelerating electric field are constant with time. The idea is

that the relativistic increase of the particle revolution period T per turn which perturbs the operation of the cyclotron can be maintained equal to T_0 without losing the synchronism between the particle revolution and the alternative electric field.

We shall describe the principle of operation of the microtron, proposed by Veksler as early as 1944 (see [18]), for the simplest case of practical significance. Let us consider a homogeneous time-independent magnetic field $B = \text{const.}$ and an accelerating gap, to which a variable voltage $V \cos 2\pi t/T_0$ is applied. The particle revolution period after k accelerations is equal to

$$T_k = \frac{2\pi E_k}{ceB}, \tag{1.57}$$

and is proportional to the total particle energy t_k for the operating condition mentioned. The increase $\Delta T_k = T_{k+1} - T_k$ is chosen in the microtron regime of operation equal to the accelerating voltage period T_0 or a multiple thereof: $\Delta T = hT_0$, h being an integer different from zero and constant for a given machine. Then, although the relativistic mass increase of the particle will cause its phase to gain progressively on the phase of the accelerating voltage, the phase difference between particle and voltage will remain a multiple of 2π. Thus one obtains a resonant mode of acceleration the condition for which can be written

$$\frac{2\pi V \cos \varphi_s}{cB} = hT_0, \tag{1.58}$$

where φ_s is the equilibrium phase. It follows that when T_0 and B are given the voltage amplitude V cannot be chosen at will but should satisfy the condition

$$\frac{V}{\lambda} > \frac{hB}{2\pi} \quad \text{or } B \text{ (kilocycles)} \lesssim \frac{20V \text{ (megavolt)}}{h\lambda \text{ (centimetres)}} \tag{1.59}$$

where $\lambda = cT_0$ is the wave-length of the accelerating voltage. This last requirement determines the constructional peculiarities of the accelerator. As it is very inconvenient to use magnets with small induction values B, one must take quantities V/λ of the order of at least tens or hundreds of kilovolts per centimetre. One must employ centimetre-wave

generators (microwaves), hence the name microtron*. Evidently the coefficient h in (1.58) and (1.59) should be chosen as small as possible, that is $h = 1$.

In the case considered with homogeneous magnetic field the trajectories of the accelerated particles have the form shown in fig. 65. They

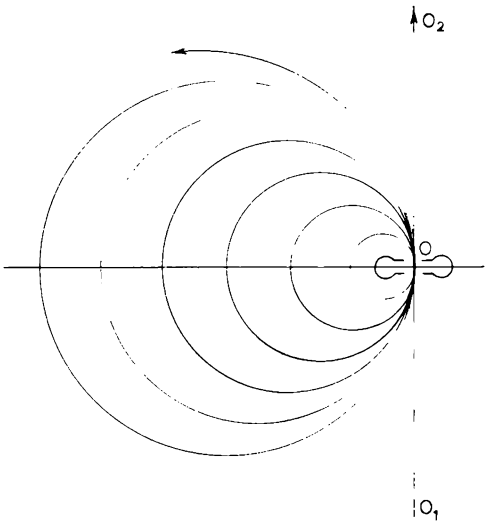

Fig. 65. Particle trajectory in a microtron with one accelerating gap.

are circles of increasing radii all going through the same point O where the accelerating gap lies. If the initial direction of flight of the particle coincides with the axis of the gap O_1O_2 along which the accelerating force is directed, this axis is a common tangent to all circles. If the trajectory makes an angle γ_i with O_1O_2 initially, this angle will decrease rapidly in the course of acceleration, being proportional to $1/p$ (p is the particle momentum). The phase motion does not depend on γ_i to a first approximation.

Condition (1.58), which determines ΔT, has to be supplemented by the initial resonance condition

$$T_i = \frac{2\pi E_i}{ceB} = q_i T_0 , \tag{1.60}$$

* Sometimes the microtron is also called electron cyclotron.

where q_i is an integer (the initial harmonic number). For the choice of the injection energy E_i this equation can be fulfilled independently of (1.58). In the case of electrons one usually chooses $q_i = 1$, $E_i = E_0 + eV \cos \varphi_s$, where E_0 is the electron rest energy. For protons, evidently, the quantity q_i has to be much larger than for electrons (in the ratio of the masses of these particles). This circumstance makes it difficult to accelerate protons in the microtron regime.

In contrast with other accelerators, where one can limit oneself to differential equations, it is convenient with the microtron to make use of finite-difference equations when investigating the phase motion in a microtron (see [19]). For functions which have integral arguments F_k ($k = 1, 2, 3, \ldots$) one introduces instead of derivatives finite differences of corresponding order: a first difference $\Delta F_k = F_{k+1} - F_k$, a second difference $\Delta^2 F_k = \Delta F_{k+1} - \Delta F_k = F_{k+2} - 2F_{k+1} + F_k$, and so on [20]. Making use of (1.57), we obtain a finite-difference equation of the second order, which is valid also for other cyclic resonance accelerators

$$\Delta^2 t_k - \frac{2\pi V}{cB} \cos \omega_0 t_{k+1} = 0 , \tag{1.61}$$

where t_k is the time of the kth crossing of the accelerating gap and where $\Delta t_k = T_k$ and $\Delta^2 t_k = \Delta T_k$. The increase of the phase during the kth revolution is evidently

$$\Delta \varphi_k = \omega_0 \Delta t_k - 2\pi q_k , \tag{1.62}$$

where q_k is the harmonic number for this revolution according to the principle of the microtron

$$\Delta q_k = h . \tag{1.63}$$

Hence, $\Delta \varphi_k$ is proportional to the difference in energy gain per turn between an equilibrium and a non-equilibrium particle, that is $\Delta \varphi_k \propto \Delta E_k - \Delta(E_k)_s$. From (1.62) and (1.63) we arrive at the phase equation we were seeking

$$\Delta^2 \varphi_k - \frac{2\pi h \cos \varphi_{k+1}}{\cos \varphi_s} = -2\pi h , \tag{1.64}$$

of which the equilibrium phase φ_s (and also $-\varphi_s$) is a particular stationary solution.

Let us stress that both the microtron and the cyclotron are systems in which one can not only accelerate but also decelerate in resonance, because B, as well as ω_0, is constant. What the particle will do depends on the initial conditions. The phase equation for deceleration is obtained from (1.64) by putting $-h$ for h and $\pi-\varphi_s$ for φ_s.

A phase equation of the type

$$\Delta^2\varphi_k - D\cos\varphi_{k+1} = -D\cos\varphi_s\,, \tag{1.65}$$

to which (1.64) belongs applies quite generally to accelerators with self-focusing. However, the quantity D is of the order of $D \approx 2\pi$ for microtrons whereas $D \ll 2\pi$ for other accelerators. The consequence of this is that the phase varies practically without discontinuities in these accelerators and one can pass from the difference equation to a differential equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}k^2} - D\cos\varphi = -D\cos\varphi_s\,, \tag{1.66}$$

which happens to be the equation of the pendulum with an external torque. In the microtron this transformation would not be legitimate because the quantity D is large and the difference and differential equations would lead to different results.

Let us study first small phase oscillations in the neighbourhood of φ_s, writing

$$\varphi_k = \varphi_s + \eta_k\,, \qquad |\eta_k| \ll \varphi_s\,. \tag{1.67}$$

We linearize (1.64) in η_k and obtain the linear difference equation

$$\eta_{k+2} - 2(1-\pi h\ \mathrm{tg}\ \varphi_s)\eta_{k+1} + \eta_k = 0\,, \tag{1.68}$$

the solution of which has the form

$$\eta_k = b\cos(\mu k+\delta)\,, \qquad \cos\mu = 1-\pi h\ \mathrm{tg}\ \varphi_s\,, \tag{1.69}$$

where b, δ are constants and μ is a quantity analogous to the characteristic number introduced in ch. 2 §§ 3 and 4. As appears from (1.69), it is only when the quantities μ and φ_s are extremely small, $\varphi_s \ll 1/2\pi h$ and $\mu \approx \sqrt{(2\pi h\varphi_s)} \ll 1$, that one can describe the small phase oscillations around φ_s by the corresponding differential equation.

As follows from (1.69), stable phase motion is only possible in a range of values of φ_s given by

$$|\cos \mu| < 1 \quad \text{or} \quad 0 < \operatorname{tg} \varphi_s < \operatorname{tg}(\varphi_s)_{\text{lim}} = 2/\pi h \,. \tag{1.70}$$

The inequality on the left which states that the phase φ_s must lie in the region where the voltage decreases with time, is common to all cyclic weak-focusing accelerators (see ch. 1 § 4). The inequality on the right is less trivial and shows that in the microtron for large φ_s the self-focusing mechanism is too strong and leads to build-up of phase oscillations [19]. The existence of an upper limit for the value of φ_s is an important feature of the microtron which does not appear in any other accelerator. This upper limit $(\varphi_s)_{\text{lim}}$ is equal to 0.209 for $h = 3$, to 0.308 for $h = 2$, and to 0.587 for $h = 1$.

The optimum value $(\varphi_s)_{\text{opt}}$, which corresponds to $\mu = \frac{1}{2}\pi$, that is to the centre of the stability region, is according to (1.69) equal to $(\varphi_s)_{\text{opt}} = \operatorname{arc\,tg}(1/\pi h)$. The phase oscillation period is then $k_0 = 4$. When φ_s varies from 0 to $(\varphi_s)_{\text{lim}}$ the quantity k_0 varies from $k_0 = \infty$ to $k_0 = 2$. Outside of this range an instability occurs, the character of which depends on whether $\varphi_s \leqslant 0$ or $\varphi_s \geqslant (\varphi_s)_{\text{lim}}$. In the former case η_k grows without oscillation

$$\eta_k = b \cosh(\mu k + \delta) \,, \qquad \cosh \mu = 1 + \pi h \operatorname{tg} |\varphi_s| \,, \tag{1.71}$$

while in the second case a phase oscillation of period $k = 2$ takes place with amplitude increasing according to the law

$$\eta_k = (-1)^k b \cosh(\mu k + \delta) \,, \qquad \cosh \mu = \pi h \operatorname{tg} \varphi_s - 1 \,. \tag{1.72}$$

It is possible to illustrate the behaviour of phase oscillations for various values of φ_s or what comes to the same thing, for various values of μ ($\mu \approx 0$, $\mu \approx \frac{1}{2}\pi$, $\mu \gtrsim \pi$) by graphs similar to those which were introduced in § 2 for the strong-focusing synchrophasotron (see fig. 70).

We have seen that small linear oscillations around equilibrium phase are not damped. However, even for arbitrary values of $\eta_k = \varphi_k - \varphi_s$ there is no reason to expect a compression of the $\eta_k, \Delta\eta_k$ region with respect to its initial state. According to (1.64) one can write quite generally

$$\begin{aligned} \eta_{k+1} &= \eta_k + \Delta\eta_k \,, \\ \Delta\eta_{k+1} &= \Delta\eta_k + 2\pi h [\cos(\eta_k + \Delta\eta_k) - \operatorname{tg} \varphi_s \sin(\eta_k + \Delta\eta_k) - 1] \,. \end{aligned} \tag{1.73}$$

It is easily seen that the Jacobian of the transformation leading from $(\Delta\eta_k, \eta_k)$ to $(\Delta\eta_{k+1}, \eta_{k+1})$ is equal to unity. So the surface corresponding to the beam particles in the plane of these variables remains constant and there is no damping. This is equivalent to Liouville's theorem for the difference equations (1.64).

For large excursions from the equilibrium conditions $\varphi_i = \varphi_s$, $\Delta\varphi_i = 0$, one must plot the successive positions of the representative point on the plane $(\varphi_k, \Delta\varphi_k)$ using the non-linear difference equation (1.64) as

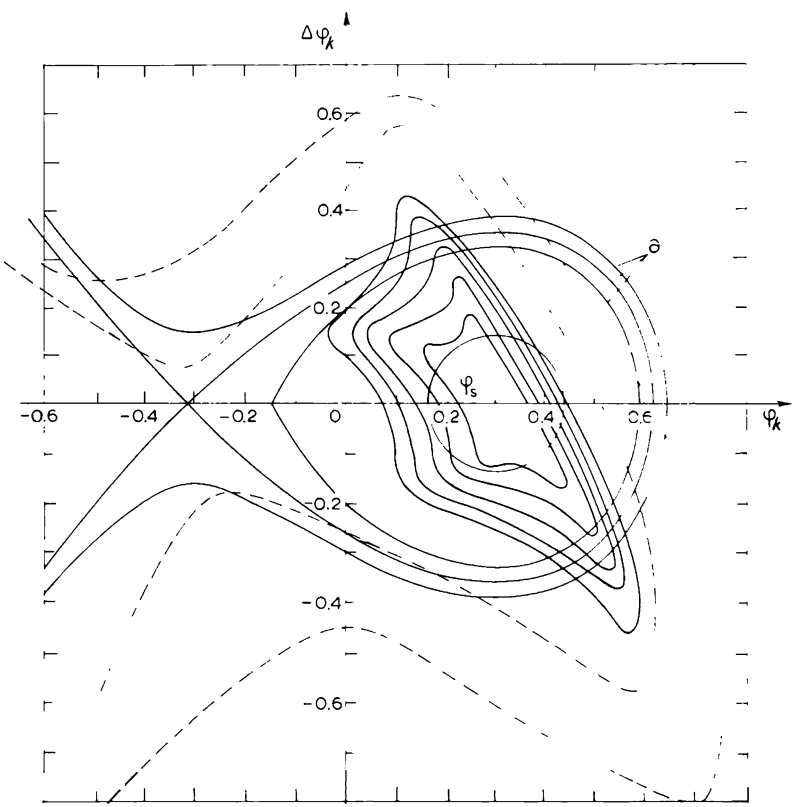

Fig. 66. Phase orbits in the case where $\varphi_s = (\varphi_s)_{opt} = 0.308$. $\hat{c}$, family of orbits described by differential equation (1.66).

a recurrence formula when going from $\varphi_k, \Delta\varphi_k$ over to $\varphi_{k+1}, \Delta\varphi_{k+1}$ after having started with $\varphi_i, \Delta\varphi_i$. The results of the numerical computation with the recurrence formula are shown in fig. 66. For initial conditions in the neighbourhood of those for equilibrium the representative point describes a motion which is nearly periodic, jumping from one position to another along a rather smooth phase trajectory, which is given by the initial values. In fig. 66 such trajectories are shown for the case $\varphi_s = (\varphi_s)_{opt} = \text{arc tg } \pi^{-1}$. The phase trajectories $d\varphi/dk = f(\varphi)$ and the bucket form of difference equation (1.66) for the same value of φ_s are

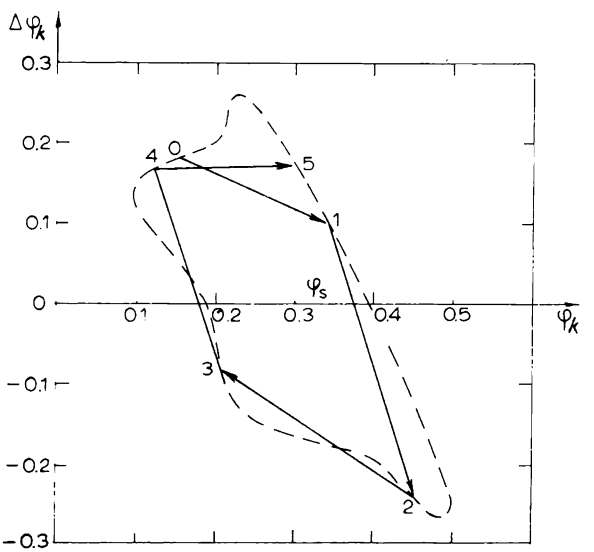

Fig. 67. Successive positions of the representative point in the phase plane for one of the phase trajectories in the case $\varphi_s = (\varphi_s)_{opt} = 0.308$ (see fig. 66).

also shown. The larger the value of φ_s the more the families of trajectories belonging to (1.64) and (1.66), respectively, will differ.

The successive positions of the representative point along the orbit in the plane $(\varphi_k, \Delta\varphi_k)$ are shown in fig. 67. If one follows this point in the course of a sufficiently large number of passages, one finds that it performs a sort of precession. One can distinguish between its basic revolution period, which is of the order of $k_0 \approx 2\pi/\mu$ and the precession

period k_{prec}. The letter is the number of positions k after which the representative point moving along the phase orbit after each revolution will come back to the neighbourhood of φ_i, $\Delta\varphi_i$. The initial excursions η_i (under the condition $\Delta\eta_i = 0$) for which k_{prec} reaches a value of the order of one can be somewhat arbitrarily taken as the limits of the phase stable region and appear to be about 0.5 rad, that is between 5 and 10% of the complete cycle.

The open phase trajectories which are dotted in fig. 66 correspond to a non-periodic motion of the representative point. This point does not even make one turn along them before leaving the neighbourhood of the equilibrium point $(\varphi_s, 0)$. Being in unstable phase conditions the particles are as likely to experience an accelerating as a decelerating field, depending whether they happen to be near the equilibrium phase φ_s or near $\pi - \varphi_s$.

To conclude let us list the advantages and disadvantages of the microtron. In contrast with all other cyclic accelerators (with the exception of the cyclotron) the capture of the particles into the acceleration mechanism takes place during each period of the high-frequency accelerating voltage, and not at relatively large intervals of time like the magnet or r.f. cycling time. Thus the mean beam intensity can, in principle, be much larger in the microtron than in the other accelerators with variable parameters such as the synchrotron or the synchrocyclotron. The microtron is particularly simple and reliable as an accelerator of electrons with a large monoenergetic beam but a relatively low energy.

An inherent disadvantage of the microtron is the fact that its parameters must be related by inequality (1.59). With the values of V and λ that are realistic at present for the accelerating cavity, the magnetic field B has to be unusually small for an accelerator (1 kgauss) and the dimensions of the solid (i.e. not annular) magnet become too large. This is why the microtron has not up to now enjoyed a widespread popularity.

There exist only a few machines in operation for electron acceleration up to energies of 5 to 10 MeV (see for instance [21]).

To achieve vertical focusing, one can in principle use azimuthal variation of the magnetic field [22, 23]. In this case one should use edge-focusing (see § 2) in a sector magnet. The sector would also permit the installation of linear-accelerator structures in the field-free straight sections, which should increase the energy gain per turn.

2. THE SYNCHROPHASOTRON

Magnets for high-energy accelerators must necessarily be annular and have relatively small radial and vertical gap dimensions, because a solid magnet of the cyclotron type would have unreasonably large dimensions and weight. Clearly, in a ring magnet with fixed orbit length Π_s, one must apply an acceleration programme whereby the frequency ω_0 of the electric field varies proportionally to the particle velocity

$$\omega_0(t) = \frac{2\pi q}{\Pi_s} v_s(t) . \tag{2.1}$$

In the case of heavy particles (protons for instance) this means a frequency change by a factor 5 or 10*. Proton accelerators in which the equilibrium orbit remains constant but ω_0 and B vary with time are usually referred to in the Soviet literature as synchrophasotrons, as they have features of both the synchrotron and the phasotron. The magnet of a large synchrophasotron is divided into separate sectors of magnetic field between which there are straight sections for installing accelerating units, vacuum pumps, extraction and injection devices, etc. In accelerators used at present the sector ends are perpendicular to the equilibrium orbit (see figs. 1 and 9). This orbit is nearly circular, consisting of arcs with a radius of curvature R aligned with straight sections of length l each. On the orbit the following relation is fulfilled at any moment

$$E^2 = E_0^2 + e^2 B^2(\vartheta, t) R^2(\vartheta) , \tag{2.2}$$

where R is the radius of curvature of the orbit at the point ϑ considered. For $R = \text{const.}$ the variations of the quantities ω_0 and B with time are therefore not independent. The frequency ω must 'follow' the increase of the magnetic field B according to a law $\omega_0 = \omega_0(B, t)$, which is given by (2.1) and (2.2),

$$\omega_0(t) = \frac{ceq\, B(t)}{(1 + L/2\pi R)\left[E_0^2 + e^2 R^2 B^2(t)\right]^{\frac{1}{2}}} , \tag{2.3}$$

* For electrons which have already nearly light velocity at injection ($\beta \lesssim 1$) the revolution frequency ω_0 is practically constant and acceleration occurs with nearly constant ω_0 (synchrotron operation).

L being the sum of the lengths of the straight sections. The ratio $L/2\pi R$ is usually equal to 0.2 to 0.3.

Relations (2.1) and (2.3) apply to strong-focusing as well as to weak-focusing synchrophasotrons. We have examined in earlier chapters the motion of the particles in a quite general manner, and for periodic systems. We have considered in detail the influence of perturbations, which is extremely important for this class of machines. To apply the results obtained and to make the necessary numerical estimates, one should have practical expressions for such characteristics of the motion of the particle as the betatron oscillation frequencies ν_x, ν_z, Floquet's functions $f_{x,z}(\vartheta)$, orbit expansion coefficient α, etc. The present section will be devoted to the derivation of these expressions.

Betatron oscillations in periodic systems are described by equations with periodic coefficients (ch. 2, eq. 2.13). In a synchrophasotron the field characteristics change little along a sector of the ideal orbit (arc of a circle) inside a magnet unit, and in a straight section between magnet units there is practically no magnetic field. Thus one may assume that the coefficients $g_{x,z}(\vartheta)$ in eq. (2.13) of ch. 2 are stepped constant functions, in other words the quantities $g_{x,z}(\vartheta)$ are equal to given constants, positive, negative, or zero on focusing (F), defocusing (D), or field-free sections of the orbit, respectively. This peculiarity of the orbits in the synchrophasotron allows an explicit presentation of the transfer characteristics of the motion.

2.1. Strong-focusing synchrophasotrons

Usually, in strong-focusing synchrotrons [24] a period consists of a focusing and a defocusing sector, between which there are straight sections*. More elaborate arrangements can also be found (see the FOFDOD system further below, and also ch. 4 § 7). We shall denote by N the number of periods as before. The azimuthal extent $2\pi/N$ of a period will be designated by $\vartheta_O = \vartheta_F + \vartheta_{O_1} + \vartheta_D + \vartheta_{O_2}$. As $g_{x,z}(\vartheta)$ has a step form, it is convenient to write down the betatron oscillations in the intervals F, O_1, D, O_2 separately, counting the azimuthal co-ordinate from the beginning of each interval. For the kth period it thus becomes

* A magnet with periods of this kind is called an FODO magnet for brevity.

possible to write down the solution of the equation of motion $y(\vartheta) = x(\vartheta)$ or $z(\vartheta)$ in the form

$$y(\vartheta) = \begin{cases} C_k^{\mathrm{F}} \cos \kappa_1 \theta + S_k^{\mathrm{F}} \sin \kappa_1 \theta\,, & (\mathrm{F}) \quad (2.4a) \\ C_k^1 + S_k^1 (\sigma / l_1) & (\mathrm{O}_1) \quad (2.4b) \\ C_k^{\mathrm{D}} \cosh x_2 \theta + S_k^{\mathrm{D}} \sinh \kappa_2 \theta\,, & (\mathrm{D}) \quad (2.4c) \\ C_k^2 + S_k^2 (\sigma / l_2)\,, & (\mathrm{O}_2) \quad (2.4d) \end{cases}$$

where

$$\theta = \sigma / R_{1,2}\,, \tag{2.5}$$

σ being the co-ordinate along the equilibrium orbit, $R_{1,2}$ the radii of curvature of this orbit in the F and D sectors, and $l_{1,2}$ the straight section lengths. The quantities $\kappa_{1,2}$ are equal to

$$(\kappa_{1,2})_x = \begin{cases} (1+n_-)^{\frac{1}{2}} \\ (n_+ - 1)^{\frac{1}{2}} \end{cases} \qquad (\kappa_{1,2})_z = \begin{cases} n_+^{\frac{1}{2}} \\ n_-^{\frac{1}{2}} \end{cases} \tag{2.6}$$

Here n_+ is for $n > 0$, whereas n_- is for the absolute value of $n < 0$. Let θ_1 and θ_2 denote the angular dimensions of the F and D sectors, respectively. The quantities $C_k^{\mathrm{F}}, S_k^{\mathrm{F}}, C_k^1, \ldots$ in (2.4) are constants within the limits of their respective intervals, but change from one period to the other. One obtains the law of this variation by 'matching' solutions of the type (2.4), that is by requiring that the trajectories should be continuous and the tangents conserved at the boundaries of the interval:

$$y_{\mathrm{left}}(\theta_{\mathrm{bound}}) = y_{\mathrm{right}}(\sigma_{\mathrm{bound}})\,, \qquad \left[\frac{\mathrm{d}y(\theta_{\mathrm{bound}})}{R\,\mathrm{d}\theta}\right]_{\mathrm{left}} = \left[\frac{\mathrm{d}y(\sigma_{\mathrm{bound}})}{\mathrm{d}\sigma}\right]_{\mathrm{right}}. \tag{2.7}$$

Here the subscripts 'right' and 'left' refer to the values at the boundary when coming from the right or from the left. Details of the calculations given further below can be found in [25], [28].

We shall now consider the coefficients $C_k^{\mathrm{F}}, S_k^{\mathrm{F}}$ as the components of the two-dimensional vectors

$$\boldsymbol{Y}_k^{\mathrm{F}} = \begin{pmatrix} C_k^{\mathrm{F}} \\ S_k^{\mathrm{F}} \end{pmatrix}, \qquad \boldsymbol{Y}_k^1 = \begin{pmatrix} C_k^1 \\ S_k^1 \end{pmatrix}. \tag{2.8}$$

To the transitions $\mathrm{F}_k \to \mathrm{O}_{1k} \to \mathrm{D}_k \to \ldots$ correspond linear transformations

of these vectors, which are performed with the help of the matrices $\mathbf{Q}^{\mathrm{F}}, \mathbf{Q}^{(1)}, \ldots$

$$\boldsymbol{Y}_k^1 = \mathbf{Q}^{\mathrm{F}} \boldsymbol{Y}_k^{\mathrm{F}}, \qquad \boldsymbol{Y}^{\mathrm{D}} = \mathbf{Q}^{(1)} \boldsymbol{Y}_k^1, \tag{2.9a}$$

$$\boldsymbol{Y}_{k+1}^{\mathrm{F}} = \mathbf{Q} \boldsymbol{Y}_k^{\mathrm{F}}, \qquad \mathbf{Q} = \mathbf{Q}^{(2)} \mathbf{Q}^{\mathrm{D}} \mathbf{Q}^{(1)} \mathbf{Q}^{\mathrm{F}}. \tag{2.9b}$$

The quantities $\boldsymbol{Y}, \mathbf{Q}$ are similar to the quantities $\boldsymbol{X}, \mathbf{M}$ seen in ch. 2 § 3. However, with stepped constant functions $g(\vartheta)$ the use of $\boldsymbol{Y}$ and $\mathbf{Q}$ is more convenient. A simpler calculation with formulae (2.4) to (2.9) yields for the component Q_{ik} the expressions

$$Q_{11} = \mathrm{c\,ch} - (\eta + \xi_1 \xi_2)\,\mathrm{s\,sh} + \xi_2\,\mathrm{c\,sh} - \xi_1 (1 + \chi^{-1})\,\mathrm{s\,ch}, \tag{2.10a}$$

$$Q_{12} = \mathrm{s\,ch} + (\eta + \xi_1 \xi_2)\,\mathrm{c\,sh} + \xi_2\,\mathrm{s\,sh} + \xi_1 (1 + \chi^{-1})\,\mathrm{c\,ch}, \tag{2.10b}$$

$$Q_{21} = -\mathrm{s\,ch} + \eta^{-1}\,\mathrm{c\,sh} - \chi \xi_2\,\mathrm{s\,sh}, \tag{2.10c}$$

$$Q_{22} = \mathrm{c\,ch} + \eta^{-1}\,\mathrm{s\,sh} + \chi \xi_2\,\mathrm{c\,sh}, \tag{2.10d}$$

in which $\mathrm{s} = \sin \kappa_1 \theta_1$, $\mathrm{sh} = \sinh \kappa_2 \theta_2, \ldots$ and

$$\eta = \frac{\kappa_1 R_2}{\kappa_2 R_1}, \qquad \xi_i = \frac{\kappa_i l_i}{R_i} \quad (i = 1, 2), \qquad \chi = \frac{l_1}{l_2}. \tag{2.11}$$

The components of each of the 'interval' matrices $\mathbf{Q}^{\mathrm{F}}, \mathbf{Q}^{\mathrm{D}}$ are obtained from (2.10) by letting the azimuthal dimensions of the remaining sections of the period tend towards zero. We look for a solution of the vector equation (2.9b) in the form

$$\boldsymbol{Y}_k^{\mathrm{F}} = \boldsymbol{G} \mathrm{e}^{\mathrm{i}\mu k} + \mathrm{c.c.}, \qquad \boldsymbol{G} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \tag{2.12}$$

where $\boldsymbol{G}$ is a constant vector, depending on the parameters of the system and on the initial conditions and μ, as in ch. 2, defines the betatron oscillation frequency $\nu = N\mu/2\pi$. The components G_1, G_2 of the vector are in general complex. From (2.9b) and (2.12) we obtain

$$\frac{G_2}{G_1} = \frac{\mathrm{e}^{\mathrm{i}\mu} - Q_{11}}{Q_{12}} = \frac{Q_{21}}{\mathrm{e}^{\mathrm{i}\mu} - Q_{22}}. \tag{2.13}$$

From the expressions (2.10) for the components Q_{ik}, it follows immediately that $\det \mathbf{Q} \equiv 1$ (see ch. 2, eq. 3.11). So we get from (2.13)

$$\cos \mu = \tfrac{1}{2} \operatorname{Sp} \mathbf{Q} = \tfrac{1}{2}(Q_{11} + Q_{22}) \tag{2.14}$$

or

$$\cos \mu = \tfrac{1}{2}\{2\mathrm{c\ ch} + (\eta^{-1} - \eta - \xi_1 \xi_2)\mathrm{s\ sh} + \xi_2(1+\chi)\mathrm{c\ sh} - \xi_1(1+\chi^{-1})\mathrm{s\ ch}\}\,. \tag{2.15}$$

In the more simple case where the straight sections are absent or negligibly small, we have

$$\cos \mu = \mathrm{c\ ch} + \frac{1-\eta^2}{2\eta}\,\mathrm{s\ sh}\,. \tag{2.16}$$

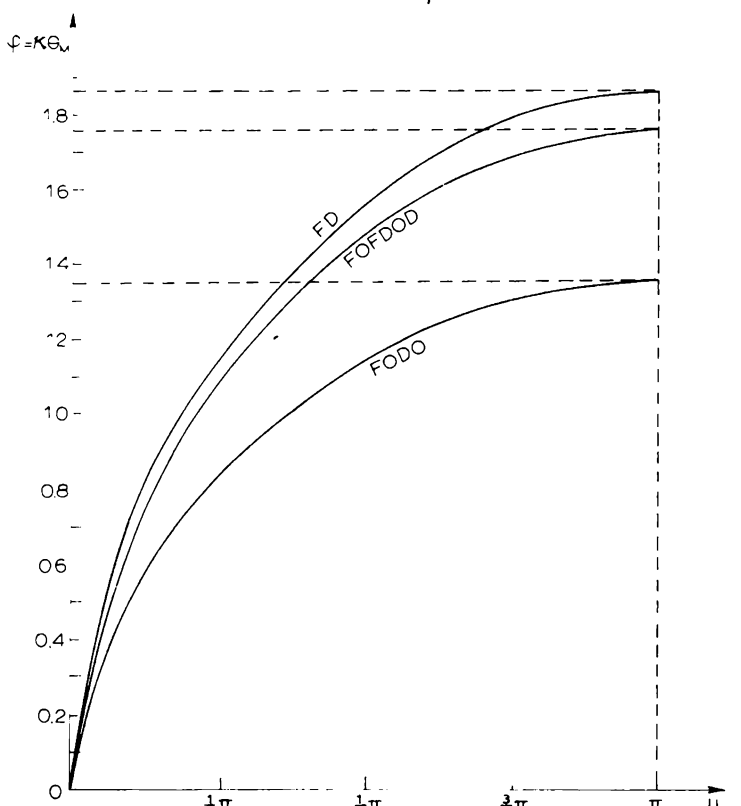

Fig. 68. Relation between the quantities μ and $\varphi = \kappa\theta_M$ for the systems FD ($\lambda = 0$), FODO and FOFDOD ($\lambda = l/R\theta_M = 0.3$).

Usually the following relations for the parameters occurring in (2.15) are valid: $R_1 \approx R_2 \approx R$, $\theta_1 \approx \theta_2 \approx \theta_M$, $l_1 \approx l_2 \approx l$, $\kappa_1 \approx \kappa_2 \approx \kappa$. Then $\cos \mu$ depends only on two parameters $\varphi = \kappa\theta_M$ and $\lambda = 1/R\theta_M$ (fig. 68). For the centre of the stability region $\mu = \frac{1}{2}\pi$ for $l = 0$, and we have $\varphi = \frac{1}{2}\pi$ or $N = 2|n|^{\pm}$. The form of the stability region is the plane $n^{\pm}_{+}\theta_M$, $n^{\pm}_{-}\theta_M$ is shown in fig. 69. As $|n|^{\pm}\theta_M$ increases, we pass through an infinity of

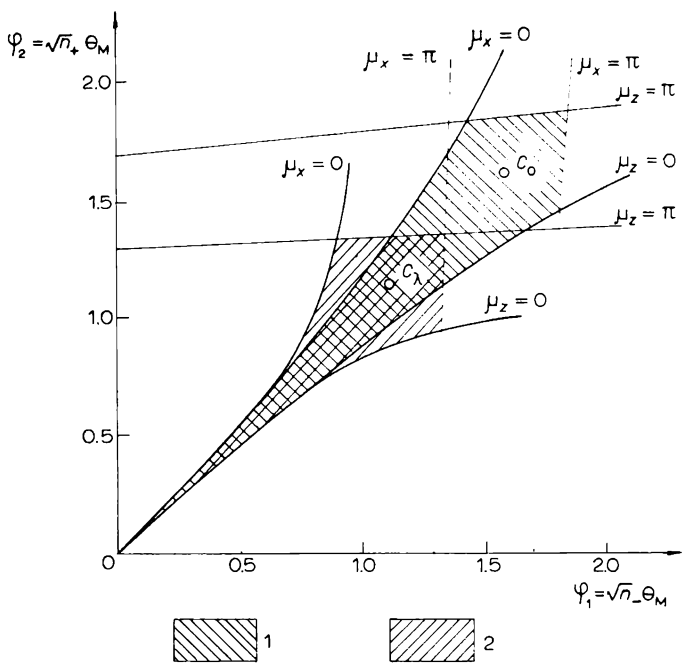

Fig. 69. The stability regions. (1) $\lambda = 0$ (2) $\lambda = 1$; C_0 and C_λ are the centres of the stability region ($\mu_{x,z} = \frac{1}{2}\pi$).

such regions, according to the general properties of Hill's equation (see ch. 2, §§ 3 and 4). The domain of stability for both x- and z-directions shown in fig. 69 is the first in the order of succession and the largest as compared to the interval $|n|^{\pm}\theta_M$. The following common stability regions are much smaller than the first one, and their dimensions decrease very rapidly.

Let us now determine the form of the betatron oscillations and their amplitude in the system considered. Evaluating from (2.4a) C_k^F, S_k^F we obtain with the help of (2.12), (2.13) the solution for $y(\vartheta)$ in the F sector in the form

$$y(\vartheta) = y(\theta) = G_1 \mathrm{e}^{\mathrm{i}\mu k} \left(\cos \kappa_1 \theta + \frac{\mathrm{e}^{\mathrm{i}\mu} - Q_{11}}{Q_{12}} \sin \kappa_1 \theta \right) + \text{c.c.} \quad (2.17)$$

Making use of the vector transformation (2.9a) and of the expressions (2.4b), (2.4c) and (2.4d) we can derive the expressions for $y(\vartheta)$ in the remaining sections. We find thus Floquet's function $f(\vartheta)$ (see ch. 2 § 4) which enters into the general solution of the betatron oscillation equation $y = a\,\mathrm{e}^{\mathrm{i}\nu\vartheta} f(\vartheta) + \text{c.c.}$

$$f(\vartheta) = \mathrm{d}\,\mathrm{e}^{\mathrm{i}\mu(k - \vartheta/\vartheta_0)} \begin{cases} \sin \kappa\theta + b \cos \kappa\theta\,, & \text{(F)} \quad (2.18a) \\ \mathrm{s} + bc + \xi(\mathrm{c} - b\mathrm{s})\sigma/l\,, & (\mathrm{O}_1) \quad (2.18b) \\ \eta(\mathrm{c} - b\mathrm{s})\ \mathrm{sh}\ \kappa\theta + \\ \quad + [(\mathrm{s} + b\mathrm{c}) + \xi(\mathrm{c} - b\mathrm{s})]\ \mathrm{ch}\ \kappa\theta\,, & \text{(D)} \quad (2.18c) \\ \mathrm{c\ sh} + \mathrm{s\ ch} + \xi\ \mathrm{c\ ch} + b(\mathrm{c\ ch} - \mathrm{s\ sh} - \\ \quad - \xi\ \mathrm{s\ ch}) + \xi(\mathrm{c\ ch} + \xi\ \mathrm{c\ sh} + \mathrm{s\ sh}) + \\ \quad + b\xi(\mathrm{c\ sh} - \mathrm{s\ ch} - \xi\ \mathrm{s\ ch})\sigma/l\,, & (\mathrm{O}_2) \quad (2.18d) \end{cases}$$

where

$$b = \frac{(2 + \xi^2)\,\mathrm{s\ sh} + 2\xi\ \mathrm{s\ ch} - 2\mathrm{i} \sin \mu}{2(-\mathrm{c\ sh} + \mathrm{s\ ch} + \xi\ \mathrm{s\ sh})}, \quad (2.19)$$

$$d^2 = \frac{\mathrm{s\ ch} + \xi\ \mathrm{s\ sh} - \mathrm{c\ sh}}{\kappa \sin \mu}, \qquad \xi = \frac{\kappa l}{R}. \quad (2.20)$$

Typical betatron trajectories are shown in fig. 70 for various positions of the operating point $(n_+^{\frac{1}{2}}\theta_M, n_-^{\frac{1}{2}}\theta_M)$. From (2.18) we find an expression for the modulus $|f(\vartheta)|$ of Floquet's function which is the envelope of the betatron oscillations:

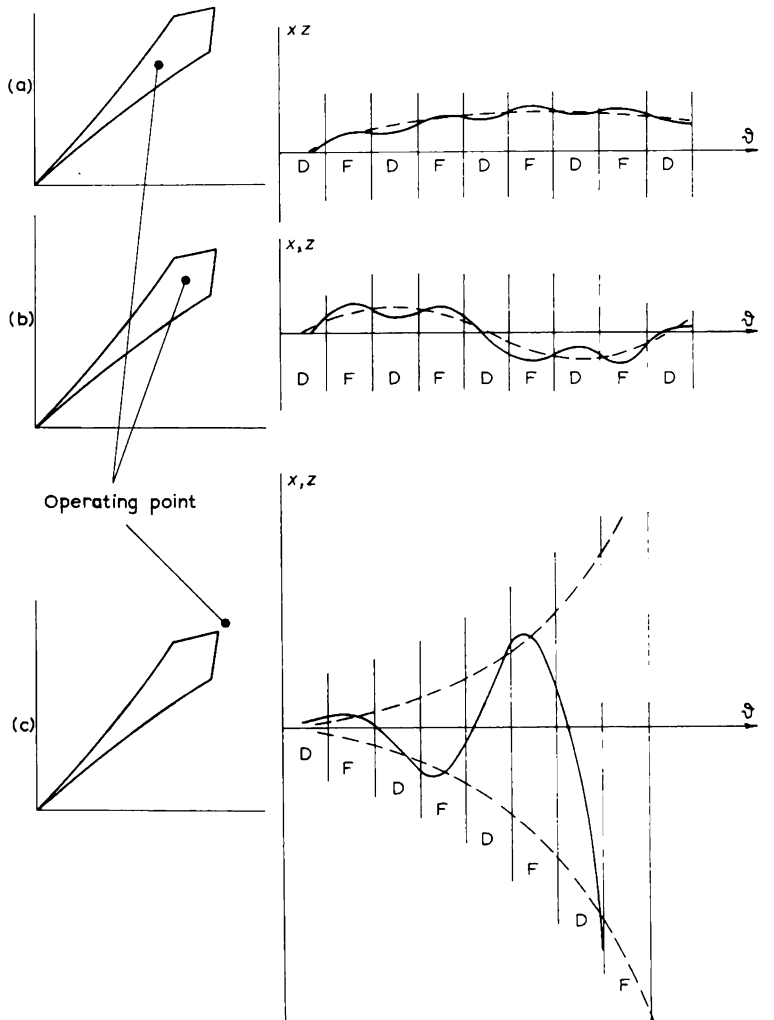

Fig. 70. Typical trajectories of particles in a strong-focusing accelerator for various positions of the operating point [26]. D and F indicate defocusing and focusing sectors. (a) The oscillation wavelength is substantially larger than the magnet period. (b) Centre of the stability region: the wave-length corresponds to four magnet periods. (c) The operating point lies outside the stability region.

$$\kappa \sin \mu |f(\vartheta)| = \begin{cases} (\mathrm{ch}+\xi\ \mathrm{sh})\mathrm{s}+[(1+\xi^2)\mathrm{sh}+2\xi\ \mathrm{ch}]+ \\ +[\xi\ \mathrm{ch}+(1+\tfrac{1}{2}\xi^2)\mathrm{sh}][\cos 2\kappa(\theta-\tfrac{1}{2}\theta_1)-\mathrm{c}], \ (\mathrm{F}) \quad (2.21a) \\ (\mathrm{ch}+\xi\ \mathrm{sh})\mathrm{s}+[2\xi\ \mathrm{ch}+(1+\xi^2)\ \mathrm{sh}]\mathrm{c}- \\ \quad -\xi^{\frac{q}{l}}\{\xi^{\frac{q}{l}}[\mathrm{s\ sh}-(\mathrm{ch}+\xi\ \mathrm{sh})\mathrm{s}]+ \\ \quad +[2\xi\ \mathrm{ch}+(2+\xi^2)\mathrm{sh}]\mathrm{s}\}\ , \ (\mathrm{O}_1) \quad (2.21b) \\ (\mathrm{c}-\xi\mathrm{s})\mathrm{sh}+[(1-\xi^2)\mathrm{s}+2\xi\mathrm{c}]\mathrm{ch}+ \\ +[\xi\mathrm{c}+(1-\tfrac{1}{2}\xi^2)\mathrm{s}][\mathrm{ch}\ 2\kappa(\theta-\tfrac{1}{2}\theta_2)-\mathrm{ch}]\ , \ (\mathrm{D}) \quad (2.21c) \\ (\mathrm{c}-\xi\mathrm{s})\mathrm{sh}+[2\xi\mathrm{c}+(1-\xi^2)\mathrm{s}]\mathrm{ch}- \\ -\xi^{\frac{q}{l}}\{\xi^{\frac{q}{l}}[-\mathrm{s\ ch}+(\mathrm{c}-\mathrm{s}\ \xi)\mathrm{sh}]- \\ \quad -[2\xi\mathrm{c}+(2-\xi^2)\mathrm{s}]\mathrm{sh}\}\ . \ (\mathrm{O}_2) \quad (2.21d) \end{cases}$$

As we see from these expressions $|f(\vartheta)|$ has its maximum in the middle of a focusing sector (at $\theta=\frac{1}{2}\theta_1$) and its minimum in the middle of a defocusing sector (at $\theta=\frac{1}{2}\theta_2$). The typical aspect of the envelope is shown in fig. 19. The quantities $P_y(\vartheta_i)_{max}=P_y(\frac{1}{2}\theta_1,\vartheta_i)$ and $P_y(\vartheta_i)_{max}=P_y(\frac{1}{2}\theta_1,\vartheta_i)$ are important characteristics of the betatron oscillations, which correspond for given initial conditions to the maximum values of the envelope in the middle of focusing sectors. The functions $P_y(\vartheta)$ and $P_y(\vartheta)$ can be determined by formula (4.14) of ch. 2. It is clear from what was said about $f(\vartheta)$ that $P_y(\vartheta_i)_{max}$ will be smallest when injection takes place in the middle of a focusing sector ($\theta_i=\frac{1}{2}\theta_1$) where the optimum angle γ_{i0} is zero. This smallest value $P_y(\vartheta_i)_{max}$ is equal to unity: $P_y(\frac{1}{2}\theta_1,\frac{1}{2}\theta_2)=1$. On the contrary, the quantity $P_y(\vartheta_i)_{max}$ will be smallest when injection takes place in the middle of a defocusing sector ($\theta_i=\frac{1}{2}\theta_2$). The smallest value $P_y(\vartheta_i)_{max}$ is then equal to $P_y(\frac{1}{2}\theta_1,\frac{1}{2}\theta_2)$. Numerical values of $P_y(\vartheta_i)_{max}$ and $M=\kappa P_y(\vartheta_i)_{max}$ are given in table 8.

It should be noted that in the expression for the maximum of the envelope $y_{max}=MR\kappa^{-1}\gamma_i$ only the factor M is new when compared with the expression for the oscillation amplitude in the case of a weak-focusing accelerator where the n (or the κ) is constant with azimuth. Consequently, for given R and γ_i the amplitude in a strong-focusing accelerator is smaller by a factor $\kappa_{strong}/M\kappa_{weak}$ than in a weak-focusing accelerator. Taking into account the numerical value of M and of κ_{weak} which is 0.6 to 0.7, we find from table 8 for $\mu=\frac{1}{3}\pi$ to $\frac{2}{3}\pi$ and $\lambda\approx 0.25$

TABLE 8

Betatron oscillations and orbit characteristics in the FODO system

μ		$\frac{1}{3}\pi$			$\frac{1}{2}\pi$			$\frac{2}{3}\pi$		
λ		0	0.25	0.5	0	0.25	0.5	0	0.25	0.5
$P_y(\vartheta_i)_{\max}\vartheta_i =$	$\theta_{F/2}$	1	1	1	1	1	1	1	1	1
	$\theta_{D/2}$	1.39	1.48	1.47	2.19	1.82	1.87	3.38	2.23	2.21
$M(\vartheta_i)\vartheta_i =$	$\theta_{F/2}$	3.96	3.83	3.95	4.81	4.30	4.55	6.42	5.73	6.09
	$\theta_{D/2}$	2.86	2.59	2.65	2.20	2.36	2.43	1.90	2.56	2.76
$\kappa^2\psi_{\max}$		8.52	6.63	5.29	6.39	5.01	3.91	5.50	4.30	3.35
$\kappa^2\alpha$		6.79	3.88	2.87	4.85	2.73	2.04	3.94	2.23	1.66
$\kappa^{-1}\alpha^{-\frac{1}{2}}$		0.38	0.51	0.59	0.45	0.60	0.70	0.50	0.67	0.78

a factor $\frac{1}{3}\kappa_{\text{strong}}$ which is of the order of 6 if $|n| \approx 300$. Such is the decrease in betatron oscillation amplitude gained when going from a weak to a strong-focusing magnet.

If the particle momentum is not equal to its equilibrium value, the closed orbit is displaced from the equilibrium position by an amount $x_0 = \psi(\vartheta)\Delta p/K_0 p$. To find the function $\psi(\vartheta)$ one can use the general formulae (1.3) of ch. 4, or apply the matrix method again (eqs. 2.4 to 2.12). The radial displacements will be described as before by formulae of the type (2.4). However, in F and D sectors (formulae 2.4a, 2.4b) they should be complemented by a term

$$x_F = \frac{R_1}{1+n_-}\frac{\Delta p}{p} \quad \text{and} \quad x_D = -\frac{R_2}{n_+ - 1}\frac{\Delta p}{p},$$

respectively. After a few calculations one obtains in the place of (2.9b) the linear non-homogeneous vector equation

$$\boldsymbol{Y}^{\mathrm{F}}_{k+1} - \mathbf{Q}\boldsymbol{Y}^{\mathrm{D}}_{k} = \boldsymbol{U}\,, \tag{2.22}$$

where the constant vector $\boldsymbol{U}$ accounts for the terms x_{F} and x_{D}. The solution of eq. (2.22) along a section of the closed orbit will be a constant vector also, in the form

$$\boldsymbol{Y}_{\mathrm{F}} = (1-\mathbf{Q})^{-1}\boldsymbol{U}\,. \tag{2.23}$$

With the help of operations of the type (2.9a) we can find the parts of the orbit corresponding to all the component parts of the magnet period and obtain the function $\psi(\vartheta)$ in the form

$$(1+\lambda)\kappa^2\psi(\vartheta) = \begin{cases} 1 + D\ \mathrm{sh}_{\frac{1}{2}}\cos\kappa(\theta-\frac{1}{2}\theta_1)\,, & (\mathrm{F}) \quad (2.24a)\\ 1 - \mathrm{D}\ \mathrm{sh}_{\frac{1}{2}}(\mathrm{s}_{\frac{1}{2}}\xi\tfrac{q}{l} - \mathrm{c}_{\frac{1}{2}})\,, & (\mathrm{O}_1) \quad (2.24b)\\ -1 + D\ \mathrm{s}_{\frac{1}{2}}\,\mathrm{ch}\,\kappa(\theta-\frac{1}{2}\theta_2)\,, & (\mathrm{D}) \quad (2.24c)\\ -1 + D\ \mathrm{s}_{\frac{1}{2}}(\mathrm{sh}_{\frac{1}{2}}\xi\tfrac{q}{l} + \mathrm{ch}_{\frac{1}{2}})\,, & (\mathrm{O}_2) \quad (2.24d) \end{cases}$$

where

$$D = \frac{2}{\mathrm{s}_{\frac{1}{2}}\,\mathrm{ch}_{\frac{1}{2}} - \mathrm{c}_{\frac{1}{2}}\,\mathrm{sh}_{\frac{1}{2}} + \xi\,\mathrm{s}_{\frac{1}{2}}\,\mathrm{sh}_{\frac{1}{2}}}\,, \tag{2.25}$$

$$\xi = \kappa l/R\,, \qquad \mathrm{s}_{\frac{1}{2}} = \sin\tfrac{1}{2}\kappa\theta_1\,, \qquad \mathrm{ch}_{\frac{1}{2}} = \mathrm{ch}\,\tfrac{1}{2}\kappa\theta_2\,,$$

The function $\kappa^2\psi(\vartheta)$ for systems consisting of F and D sectors is shown in fig. 35. Its maximum values, which are reached in the middle of F sectors, are listed in table 8. As will be seen, in the interval $\mu = \frac{1}{3}\pi$ to $\frac{2}{3}\pi$ and $\lambda = 0$ to 0.5, the quantity $(x_0)_{\max}$ will be about 4 to 8 times larger than the displacement x_{F} of the sector equilibrium circle. However, thanks to the large value of $\kappa^2_{\mathrm{strong}}$, the value of $(x_0)_{\max}$ will be substantially smaller than the orbit displacement $(x_0)_{\mathrm{weak}}$ that would have occurred in a weak-focusing magnet for the same relative momentum deviation $\Delta p/p$. One can write $(x_0)_{\max}$ in the form

$$(x_0)_{\max} = \psi_{\max}(1+\lambda)R\,\Delta p/p\,, \tag{2.26}$$

which makes it possible to compare it directly to $(x_0)_{\text{weak}}$. For given R and $\Delta p/p$ we have

$$\frac{(x_0)_{\max}}{(x_0)_{\text{weak}}} = \psi_{\max}\kappa^2_{\text{weak}}(1+\lambda)\,. \tag{2.27}$$

This means, for the centre of the stability region, that $(x_0)_{\max}$ will be smaller than $(x_0)_{\text{weak}}$ by a factor of about $0.5\kappa^2_{\text{strong}}$. For $|n| = \kappa^2 \approx 300$ the factor obtained thus is 150.

The behaviour of the perturbed closed orbit has a strong influence on the features of the synchrotron oscillations, which depend mainly on the orbit expansion coefficient α (see ch. 4, eq. 1.10). Taking $\psi(\vartheta)$ from formula (2.24) we obtain for α the expression

$$\alpha = \frac{4\mathrm{s}_{\frac{1}{2}}\,\mathrm{sh}_{\frac{1}{2}}}{\kappa^3\theta_{\mathrm{M}}(1+l/R\theta_{\mathrm{M}})\sin^2\frac{1}{2}\mu}\left(\mathrm{s}_{\frac{1}{2}}\,\mathrm{ch}_{\frac{1}{2}} + \mathrm{c}_{\frac{1}{2}}\,\mathrm{sh}_{\frac{1}{2}} + \frac{\kappa l}{R}\,\mathrm{c}_{\frac{1}{2}}\,\mathrm{ch}_{\frac{1}{2}}\right). \tag{2.28}$$

It will be seen from this that $\alpha \propto |n|^{-1}$, since $\kappa\theta_{\mathrm{M}} \approx 1$. As $|n|$ is large with strong focusing (of the order of 200 to 400), α will be small, much smaller than with weak focusing, where it is $(1-n)^{-1} \approx 3$. For instance, in the centre of the stability region, where $\kappa\theta_{\mathrm{M}} = \mu = \frac{1}{2}\pi$ $(l=0)$ we will have

$$\alpha = \frac{1}{|n|}\,\frac{8}{(\pi\ \mathrm{ctgh}\,\frac{1}{4}\pi - 1)} \approx \frac{4.85}{|n|}\,. \tag{2.29}$$

The appearance of a critical energy, caused by the smallness of α, has been examined in detail in ch. 4 § 7.

As well as the periodic magnet system we have just examined, in which the straight sections lie between F and D sectors (FODO system), a more intricate system is sometimes used, in which these sections lie in the middle of an F or a D sector (FOFDOD system). This structure consists of $\frac{\mathrm{F}}{2}\frac{\mathrm{D}}{2}$ and $\frac{\mathrm{D}}{2}\frac{\mathrm{F}}{2}$ magnet units and forms a period built up as follows: $\frac{\mathrm{F}}{2}\,\mathrm{O}\,\frac{\mathrm{F}}{2}\frac{\mathrm{D}}{2}\,\mathrm{O}\,\frac{\mathrm{D}}{2}$, where O stands for the straight section. To find the values of $\cos\mu$ and the Floquet function $f(\vartheta)$ for the FOFDOD system we shall apply the matrix method we have already used for the FODO system. The expression for $\cos\mu$ in the FOFDOD system has the form

$$\cos\mu = \mathrm{c\,ch} + \tfrac{1}{2}\xi(\mathrm{c\,sh} - \mathrm{s\,ch} + \mathrm{sh} - \mathrm{s}) + \tfrac{1}{4}\xi^2(\mathrm{ch} - \mathrm{c} - \mathrm{s\,sh})\,, \tag{2.30}$$

where $\xi = \kappa l/R \quad c = \cos \kappa\theta_M, \quad ch = ch\ \kappa\theta_M,$ (2.31)

The dependence of μ on $\kappa\theta_M$ for the FOFDOD system is shown in fig. 68 for $\lambda = l/R\theta_M = 0.3$.

As the straight sections in the FOFDOD system lie at the points where the envelope of the oscillations has its extreme values (middle of F or D sectors), they have less influence on the oscillation amplitude than in the FODO system. Let us consider two magnets: one FOFDOD, the other FODO. Let both have the same radius of curvature R in the magnet sectors and the same orbit lengths $R\theta_M$ in the sectors, and l in the straight sections. This means they have also the same parameter $\lambda = l/R\theta_M$. The equality of the angular dimensions θ_M presupposes the equality of the number of periods N. Let us also assume that the two systems have the same characteristic number μ or, because N is the same number of betatron oscillations per turn ν. This means that the tolerance requirements which are mainly based on the ν-value, will be about the same. Under these conditions the two magnets will have different values $\varphi = \kappa\theta_M = \varphi(\mu)$ and as θ_M is equal for both, different κ-values. Let us now compare the characteristic quantities $(P_y)_{max}$, $(P_\gamma)_{max}$, $(x_0)_{max}$ computed for the two systems FOFDOD and FODO, respectively. We have, according to the definitions and assumptions made

$$\frac{[(P_\gamma)_{max}]_{FOFDOD}}{[(P_\gamma)_{max}]_{FODO}} = \frac{\kappa_{FODO} M_{FOFDOD}}{\kappa_{FOFDOD} M_{FODO}} \tag{2.32a}$$

$$\frac{[(x_0)_{max}]_{FOFDOD}}{[(x_0)_{max}]_{FODO}} = \frac{\kappa^2_{FODO}(\psi_{max})_{FOFDOD}}{\kappa^2_{FOFDOD}(\psi_{max})_{FODO}} \tag{2.32b}$$

where M and $\kappa^2\psi_{max}$ are numerical factors listed for the FODO system in table 8. The ratios defined in (2.32) as well as $\frac{[(P_y)_{max}]_{FOFDOD}}{[(P_y)_{max}]_{FODO}}$ are given in table 9 for the case $\mu = \frac{1}{2}\pi$ ($\lambda = 0.25$ and 0.5). As we see, for the rather sophisticated assumptions we have made, the quantity $(P_\gamma)_{max}$ is somewhat smaller, and the quantity $(x_0)_{max}$ is much smaller in the system FOFDOD than in the system FODO. The less important quantity $(P_y)_{max}$ is larger, however, in the FOFDOD than in the FODO system.

TABLE 9

Comparison of the characteristics of betatron oscillations and orbits in the FOFDOD and FODO systems for $\mu = \frac{1}{2}\pi$

	$\lambda = 0.25$		$\lambda = 0.50$	
ϑ_i	$\theta_{F/2}$	$\theta_{D/2}$	$\theta_{F.2}$	$\theta_{D/2}$
$\frac{[(P_y(\vartheta_i))_{max}]_{FOFDOD}}{[(P_y(\vartheta_i))_{max}]_{FODO}}$	0.99	0.88	0.96	0.84
$\frac{[(P_y(\vartheta_i))_{max}]_{FOFDOD}}{[(P_y(\vartheta_i))_{max}]_{FODO}}$	1	1.15	1	1.14
$\frac{[(x_0)_{max}]_{FOFDOD}}{[(x_0)_{max}]_{FODO}}$	0.62		0.50	

An important feature of the strong-focusing machine is the spacing between neighbouring resonance values $|n_{res}|$. From expression (2.16) for $\cos \mu$ we find that the distance in terms of μ between two adjacent resonances $\Delta\mu = \pi/N$ corresponds to a relative change in n of about

$$\frac{\Delta n_{res}}{|n|} \approx \frac{u_{res}(\varphi)}{N}, \qquad u_{res}(\varphi) = \frac{2\pi \sin \mu(\varphi)}{\varphi(\text{s ch} - \text{c sh})}, \tag{2.33}$$

where $\varphi = \kappa\theta_M$. For instance, at the centre of the stability region ($\mu = \varphi = \frac{1}{2}\pi$) for $l = 0$ the factor μ_{res} defined above has a value of 2.6, so that $N \approx 50$ the ratio $\Delta n_{res}/|n|$ is equal to 3%. The tolerance for a systematic error in n should be less than Δn_{res}. One can increase Δn_{res}, that is relax the tolerances, by going either to smaller values of μ (for instance one finds $u_{res} \approx 4$ for $\mu \approx \frac{1}{4}\pi$) or to smaller N. Both ways, however, result in a decrease of the focusing force, that is an increase of the betatron oscillation amplitudes (see ch. 2, § 4).

We shall not treat here the effect of various perturbations and the method of determining the tolerances, as these questions were looked at in detail in ch. 3. We shall only give in table 10 the values of the basic parameters of the CERN 30 GeV strong-focusing synchrophasotron [26].

TABLE 10
Basic parameters of the CERN strong-focusing synchrophasotron (Geneva, Switzerland)

Maximum attainable proton energy	29.5 GeV
Radius of curvature of the orbit	70.079 m
Mean orbit radius	100 m
Number of periods	50
Magnet system	FOFDOD
Field index	288.4
Dimensions of vacuum chamber cross-section	70×145 mm^2
Betatron oscillation frequency	6.25 Mc/s
Straight sections	80 of 1.60 m
	20 of 3.0 m
Magnet weight	3200 t
Coil weight (aluminium)	110 t
Rise-time of magnetic field up to 12 kgauss	1.0 sec
Number of cycles per minute	20
Energy stored in the magnetic field	10 MJ
Accuracy in magnet position, horizontal	0.3 mm
vertical	0.6 mm
Perturbation of field index	$\pm 1\%$
Energy gain per turn	54 keV
Number of accelerating cavities	16
Harmonic number	20
Frequency variation	2.9 to 9.55 Mc/s
Critical energy	5.7 GeV
Injection energy	50 ± 0.1 MeV
Injection time	6 μs
Residual gas pressure in tank	10^{-5} mm Hg
Intensity obtained	$\approx 10^{11}$ particles per pulse

2.2. *The weak-focusing synchrophasotron*

The weak-focusing synchrophasotron, often called the race-track*, consists of N periods each of which has a magnetic-field sector and a straight section. The magnetic field in each sector has a radial gradient which is nearly constant (independent of azimuth) and rather small.

* The English name 'race-track' has its origin in the fact that the first project of this type [27] had two straight sections ($N = 2$) only, which gave it the form of a horse-racing track. It soon became clear that a magnet where $N = 2$ is not good enough for betatron oscillation stability. This is why as a rule one takes 4 or sometimes 8 straight sections for such machines, but the name race-track has survived by tradition.

When studying the betatron oscillations one can, as in the preceding section, consider that the coefficients $g(\vartheta)$ in eqs. (2.13) of ch. 2 are stepped constant functions: $g(\vartheta) \equiv \text{const.} \neq 0$ in the magnet sector, and $g(\vartheta) \equiv 0$ in the straight section. In the magnetic-field region $y(\vartheta) = x(\vartheta)$ or $z(\vartheta)$ will be expressed by harmonic functions

$$y(\vartheta) = y(\theta) = C_k^M \cos \kappa\theta + S_k^M \sin \kappa\theta \,, \tag{2.34}$$

or

$$\kappa = \begin{cases} (1-n)^{\frac{1}{2}} & \text{for the } x\text{-motion} \,, \\ n^{\frac{1}{2}} & \text{for the } z\text{-motion} \,. \end{cases} \tag{2.35}$$

Here the kth period is considered, θ being the angle in a polar co-ordinate system having its centre at the centre of the sector. Thus $0 \leqslant \theta \leqslant \theta_M$, $\theta_M = 2\pi/N$ being the angular dimension of the sector (see fig. 1). In the straight section we have

$$y(\vartheta) = y(\sigma) = C_k^0 + S_k^0(\sigma/l) \,, \tag{2.36}$$

where σ is the length along the trajectory from the beginning of the straight section ($0 \leqslant \sigma \leqslant l$).

Let us introduce as before vectors $\boldsymbol{Y}_k^M$, $\boldsymbol{Y}_k^0$, and matrices $\mathbf{Q}^M$, $\mathbf{Q}^0$, $\mathbf{Q}$ and perform the calculation by the method of 'matching'. So we find

$$\mathbf{Q} = \begin{pmatrix} \mathrm{c} - \dfrac{\kappa l}{R}\mathrm{s} & \mathrm{s} + \dfrac{\kappa l}{R}\mathrm{c} \\ -\mathrm{s} & \mathrm{c} \end{pmatrix}. \tag{2.37}$$

where $\mathrm{s} = \sin \kappa\theta_M$, $\mathrm{c} = \cos \kappa\theta_M$. Making use of formula (2.14) for $\cos \mu$ we find that in our case

$$\cos \mu = \mathrm{c} - h\mathrm{s} \,, \tag{2.38}$$

where the parameter h is equal to

$$h = \begin{cases} h_x = \dfrac{l(1-n)^{\frac{1}{2}}}{2R} \,, \\ h_z = \dfrac{ln^{\frac{1}{2}}}{2R} \,. \end{cases} \tag{2.39}$$

The quantity h is relatively small in all known machines (for the cosmotron $h_x = 0.11$, $h_z = 0.14$, the bevatron $h_x = 0.13$, $h_z = 0.16$, the JINS phasotron $h_x = 0.08$, $h_z = 0.12$). So we can take for μ the approximation valid for $h \ll 1$

$$\mu = \kappa\theta_M + h - \frac{c}{2s} h^2 + \left(\frac{1}{6} + \frac{c^2}{2s^2}\right) h^3 + \ldots \tag{2.40}$$

Already the term linear in h gives an accuracy of the order of one per cent. The quadratic term in h^2 will increase the precision a hundred times.

For stability of the x- and z-oscillations it is first of all necessary that the condition

$$0 < n < 1 , \tag{2.41}$$

should be fulfilled, which is also evident from the expressions for $h_{x,z}, \mu_{x,z}$. However, this is not sufficient. One must also satisfy the inequalities

$$|\cos \mu_{x,z}| < 1 , \tag{2.42}$$

since if (2.41) is fulfilled, the stability can still be destroyed at one of the two limits of the range defined in (2.42), actually at $\mu_{x,z} = \pi$. The requirements (2.42), associated with (2.38), set definite limits to the length of the straight section

$$l/R < 2n^{-\frac{1}{2}} \operatorname{ctg} \tfrac{1}{2}\kappa\theta_M , \qquad l/R < 2(1-n)^{-\frac{1}{2}} \operatorname{ctg} \tfrac{1}{2}\kappa\theta_M . \tag{2.43}$$

This limitation has practical sense only when there are two straight sections ($N = 2$, see fig. 71). For larger N the largest permissible value $(l/R)_{\text{lim}}$ is greater than one. For instance, with typical figures $n = \frac{2}{3}$, $N = 4$, we find $(l/R) \approx 2$ whereas we have in practice $l/R \approx 0.3$. The values of $(l/R)_{\text{lim}}$ mentioned are for vertical oscillations; for radial oscillations they are still larger with the same n (about five times). As regards the radial motion a more important (although not severe) limitation on l/R is set by the stability of synchrotron oscillations, that is the condition for the absence of a critical energy (see further eq. 2.51). In figs. 71 and 72 for the two cases $N = 2$ and 4 are shown three curves, which give the limits set to l/R by betatron x- and z-oscillations and by synchrotron oscillations for different values of n.

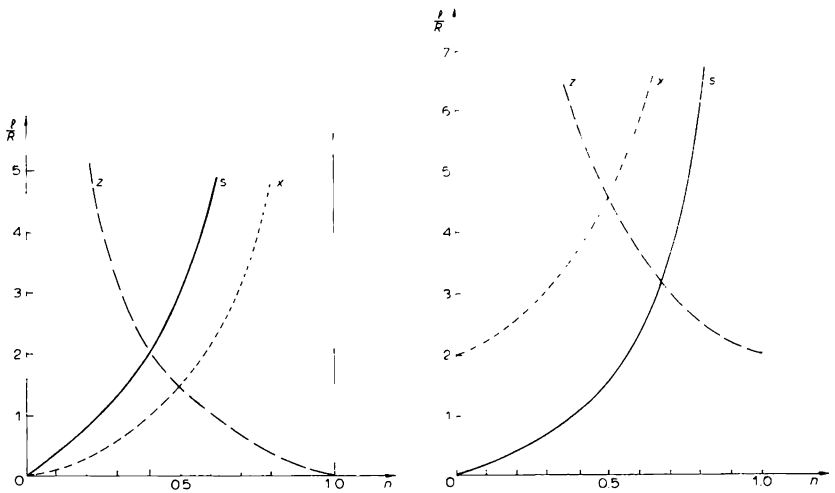

Fig. 71. Limits of the stability region for a race-track with two straight sections ($N = 2$). Curves x and z refer to betatron oscillations. Left from curve s lies the region where there is a critical energy ($\alpha < 1$).

Fig. 72. The same as fig. 71, but for $N = 4$.

We derive the expressions for Floquet's functions $f_{x,z}(\vartheta)$ and for their absolute values, which define the envelope of the betatron oscillations, by the same method as in the preceding section (see eqs. 2.4 to 2.21)

$$|f(\theta)|^2 = \frac{s + 2h \cos \kappa\theta \cos \kappa(\theta - \theta_M)}{\kappa \sin \mu} \quad \text{in a magnet sector,} \quad (2.44)$$

$$|f(\sigma)|^2 = \frac{s + 2hc + 4h^2 s \frac{\sigma}{l}(\frac{\sigma}{l} - 1)}{\kappa \sin \mu} \quad \text{in a straight section.} \quad (2.45)$$

As would be expected for reasons of symmetry, $|f(\vartheta)|^2$ is symmetrical with respect to the centre of the sections M and O. The highest value of $|f(\vartheta)|^2$ is reached in the middle of the magnet sector (for $\theta = \frac{1}{2}\theta_M$) and the smallest in the middle of the straight section (for $\sigma = \frac{1}{2}l$). As h is usually quite small, the coefficient of amplitude 'modulation' $|f|_{max}/|f|_{min}$ will exceed unity by only a few per cent in all practical cases. Examples of envelopes $|f(\vartheta)|$ are shown in fig. 73 for weak-focusing synchrophasotrons with straight sections.

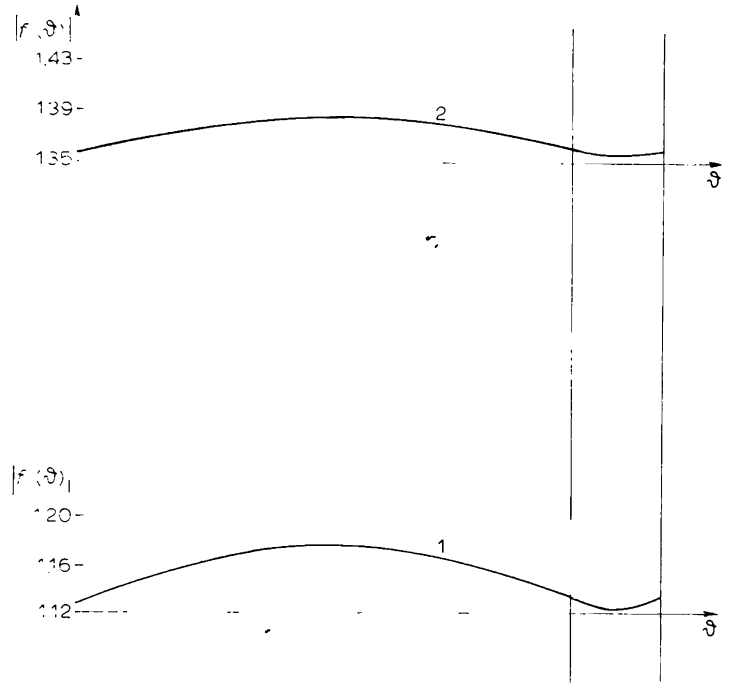

Fig. 73. Envelopes $|f(\vartheta)|$ of the betatron oscillations in the JINS synchrophasotron: $N = 4$, $n = \frac{2}{5}$, $l/R = \frac{2}{7}$. (1) for vertical, (2) for horizontal oscillations.

Thus betatron oscillations in existing proton-synchrotrons with relatively short straight sections l do not differ very much from oscillations in azimuthally symmetrical magnetic fields. This is only true however for the motion in an ideal system. If one takes perturbations into account it appears that even short straight sections can alter the picture of the motion compared with the case of azimuthal symmetry (see [28, 29]).

Let us now find the values of n for which dangerous resonances of the following types occur:

1. external resonance combined with a parametric one ($\mu_{x,z} = k\theta_M$),
2. parametric resonance $\mu_{x,z} = (k \pm \frac{1}{2})\theta_M$,
3. difference resonance $\mu_x - \mu_z = k\theta_M$, k integer,
4. sum resonance $\mu_x + \mu_z = k\theta_M$, k integer.

Making use of the expression for μ and taking condition (2.41) into account one will easily realize that in practice when the parameter ε is sufficiently small

$$\varepsilon = \frac{Nl}{4\pi R} \ll 1 , \tag{2.46}$$

the following cases can occur:

1. $k = 1$
2. $k = 2$
3. $k = 0$; $k = \pm 1$. Case 4 is impossible.

On the assumption (2.46) one can obtain the following approximate expressions for the resonance values n_{res} from formula (2.38) for μ:

$$\begin{aligned} 1.\ (n_z)_{ext} &= 1 - 2\varepsilon + (3 + \theta_M \operatorname{ctg} \theta_M)\varepsilon^2 - \\ &\quad - [2(1 + \theta_M \operatorname{ctg} \theta_M) - \tfrac{1}{3}\theta_M^2]\varepsilon^2 + \dots , \\ (n_x)_{ext} &= 1 - (n_z)_{ext} ; \end{aligned} \tag{2.47a}$$

$$\begin{aligned} 2.\ (n_z)_{param} &= \tfrac{1}{4} - \tfrac{1}{2}\varepsilon + \tfrac{1}{4}(3 + \tfrac{1}{2}\theta_M \operatorname{ctg} \tfrac{1}{2}\theta_M)\varepsilon^2 - \\ &\quad - 4(1 + \tfrac{1}{2}\theta_M \operatorname{ctg} \tfrac{1}{2}\theta_M)\varepsilon^3 + \dots , \\ (n_x)_{param} &= 1 - (n_z)_{param} ; \end{aligned} \tag{2.47b}$$

$$\begin{aligned} 3.\ (n_0)_{diff} &= \tfrac{1}{2} , \\ (n_1)_{diff} &= \varepsilon^2 - (3 + \theta_M \operatorname{ctg} \theta_M)\varepsilon^3 + \dots , \\ (n_{-1})_{diff} &= 1 - (n_1)_{diff} . \end{aligned} \tag{2.47c}$$

We shall expand n_{res} in powers of ε for two special cases

$$\begin{aligned} N = 4: \quad & 1)\ n_z = 1 - 2\varepsilon + 3\varepsilon^2 - 2.35\varepsilon^3 + \dots ; \\ & 2)\ n_x = 0.75 + 0.5\varepsilon - 0.95\varepsilon^2 + 1.68\varepsilon^3 + \dots , \\ & 3)\ n_1 = \varepsilon^2 - 3\varepsilon^3 + \dots ; \end{aligned} \tag{2.48a}$$

$$\begin{aligned} N = 8: \quad & 1)\ n_z = 1 - 2\varepsilon + 3.78\varepsilon^2 - 6.73\varepsilon^3 + \dots ; \\ & 2)\ n_x = 0.75 + 0.5\varepsilon - 0.99\varepsilon^2 + 1.92\varepsilon^3 + \dots ; \\ & 3)\ n_1 = \varepsilon^2 - 3.78\varepsilon^3 + \dots . \end{aligned} \tag{2.48b}$$

Resonance values of n for the JINS synchrophasotron, the bevatron and the cosmotron are listed in table 11. The 'position' of the resonances can be clearly seen in fig. 74, where the lines corresponding to the linear resonances of type 1, 2 and 3 are drawn in the plane $(n, l/R)$ for the case $N = 4$.

From what we have said above it is clear that even with relatively short straight sections the change in the resonance values n_{res} compared with the azimuthally symmetrical field can be substantial in practical cases. The most dangerous external resonance (together with the parametric one) which for $l = 0$ lies at the boundaries of the stability region $(n_z = 1, n_x = 0)$ is displaced when there are straight sections and appears for values like $n_z \approx 0.8$, $n_x \approx 0.2$.

The parametric resonance, corresponding to the second harmonic of the perturbation of the magnetic field index n, is slightly displaced in accelerators with straight sections towards the boundaries $n = 0$, $n = 1$ compared with the case of azimuthal symmetry. So n_x takes for instance the value 0.8 instead of 0.75. In practical machines the n-values for the parametric resonance in the radial direction n_x happen to be about equal to the n_z-values for the external resonance in the vertical direction $(n_x)_{\text{param}} \approx (n_z)_{\text{ext}}$ and vice-versa.

The sum resonance points, which lie on the boundaries $n = 0$, $n = 1$ for azimuthal symmetry, move (although very little) across these limits when there are straight sections.

The two boundary difference resonance points, on the other hand, enter the stability region by a small amount of the order of ε^2. The mid-point $n = \frac{1}{2}$, which corresponds to equal betatron frequencies in the x- and z-directions, remains where it was.

Those perturbations which distort the magnetic median plane are especially dangerous, as the space available for the motion in the vertical direction is much smaller than in the radial direction. In particular, the distortion associated with the first harmonic is proportional to the resonance factor $[n - (n_z)_{\text{ext}}]^{-1}$ (see ch. 2 § 3). As n is chosen in the range from 0.6 to 0.7 and $(n_z)_{\text{ext}}$ is equal to 0.8 a change in n of say 0.1 could increase the excursion of the orbit considerably. Hereby the space available for vertical oscillations is reduced by $2(z_0)_{\max}$, where $(z_0)_{\max}$ is the maximum vertical displacement of the perturbed orbit (see ch. 1 § 4).

TABLE 11

Resonance values and some other parameters of three large weak-focusing synchrophasotrons

Accelerator	N	l/R	n	ε	External resonance		Parametric resonance		Coupling resonance
					n_x	n_z	n_x	n_z	n_1
Synchrophasotron (Dubna)	4	0.29	0.67	0.09	0.16	0.84	0.79	0.21	0.006
Bevatron (Berkeley)	4	0.40	0.6	0.13	0.21	0.79	0.80	0.20	0.010
Cosmotron (Brookhaven)	4	0.36	0.6	0.11	0.19	0.81	0.80	0.20	0.009

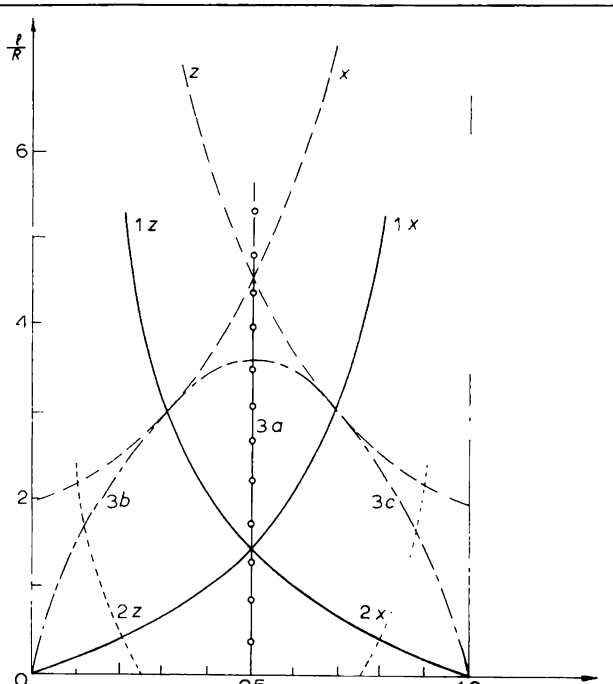

Fig. 74. Curves corresponding to linear resonances in the plane $(n, l/R)$ for $N = 4$. Curves $1x$, $1z$: external resonance with x- and z-oscillations; curves $2x$, $2z$: parametric resonance for the same; $3a$, $3b$, $3c$: difference resonances ($\nu_x - \nu_z = 0, 1, -1$).

As an example let us examine the perturbations caused in an accelerator with $N=4$ by a vertical displacement of the median plane of the magnetic sectors (quadrants); (a) the sectors are displaced in pairs in opposite directions from the median plane by a quantity δ: $\Delta z_1=\Delta z_2=\delta$ and $\Delta z_3=\Delta z_4=-\delta$ (b) the sectors are displaced one after the other in opposite directions from the median plane by the quantity δ: $\Delta z_1=\delta$, $\Delta z_2=-\delta$, $\Delta z_3=\delta$, $\Delta z_4=-\delta$. With the parameters of the JINS synchrophasotron we obtain for these cases

$$\text{(a)}\quad (z_0)_{\max}\approx\frac{\delta}{0.84-n}\approx 6\delta\ ;\qquad \text{(b)}\quad (z_0)_{\max}\approx\frac{1.6\delta}{1.84-n}\approx 1.3\delta\ .$$

One sees that with the normal value $n=0.67$ the quantity $(z_0)_{\max}$ in case (a) turns out to be six times larger than the local displacement δ of the median plane. If n has not its nominal value and comes near $n=0.84$, then $(z_0)_{\max}$ will become infinite, which corresponds to a resonance between the vertical oscillations and the first harmonic of the median plane. The situation is different in case (b) where the perturbation corresponding to the second harmonic is not in resonance and leads to a value $(z_0)_{\max}$ about equal to δ.

As the value of n changes with radius, the particles can pass once or several times through $(n_z)_{\text{ext}}$ and other resonances (especially during injection and capture into the acceleration regime, but also during extraction). Knowing the dependence $n=n(r)$ and the speed of displacement of the orbits one can find the result of these traversals by using the formulae of ch. 3 § 8.

To conclude, let us consider a few peculiarities of the behaviour of the orbits when the particle momentum p increases. The orbits are made up of straight sections of the same length l and of arcs of circle, the radius of which experiences an increase $\Delta R=R\Delta p/(1-n)p$ (see ch. 1 § 4). It follows that the function ψ is constant and that the orbit expansion coefficient α is equal to

$$\alpha=\psi=\frac{1}{(1+Nl/2\pi R)(1-n)}\,. \tag{2.49}$$

It will be seen that in principle there can exist a critical energy in the system considered $E_{\text{cr}}=E_0\alpha^{-\frac{1}{2}}$ (see ch. 4 § 7). In order that it should

not exist, the condition $\alpha > 1$ should be fulfilled or, according to (2.49),

$$\frac{l}{R} < \frac{2\pi n}{N(1-n)}. \tag{2.50}$$

In the case of a homogeneous field ($n = 0$) this condition is not fulfilled even for a very small straight section length l. However, in practice, with $n \approx \frac{2}{3}$, $N = 4$, the quantity l/R never approaches the limit defined by the inequality (2.50) and there is no critical energy as E_{cr} is smaller than E_0. For the parameters given above, the limit is $l/R \gtrsim \pi$, which is in fact much larger than the usual value $l/R \approx 0.3$ (see fig. 72).

According to ch. 4 § 4, if the condition

$$n > \frac{2}{3} + \frac{Nl}{6\pi R(1 + Nl/2\pi R)} \tag{2.51}$$

is satisfied the amplitude A_η of the phase oscillations will decrease from the beginning of the acceleration process. When this condition is not satisfied A_η will increase at the beginning up to an energy

$$E_1 = \sqrt{3} E_{cr} = [3(1-n)(1 + Nl/2\pi R)]^{\frac{1}{2}} E_0, \tag{2.52}$$

and will decrease later, when $E > E_1$. In cases met in practice this increase of A_η is not large.

Edge effects. Let us dwell briefly on the physical picture of the edge effects which appear at the boundaries between the magnet sectors and the field-free sections [25]. Generally, the end of the magnet sectors (more precisely of their stray fields) does not coincide with the direction of the radius of curvature of the orbit at the sector – straight section boundary. This leads to what is called 'edge-focusing' which causes the frequencies $\nu_{x,z}$ to change. Although this variation may be small it must be taken into account when studying questions related to stability or resonances.

Let us consider the motion in the plane $z = 0$ when entering a sector. Let its edge make an angle γ with the direction of the radius of curvature. The increase in the azimuthal co-ordinate described by the particle in the field region when entering the magnet (or leaving it) is, for $z = 0$,

$$\chi_x = Kx \operatorname{tg} \gamma. \tag{2.53}$$

The vertical motion is subject to the action of the fringing field at the end of the magnet sector end. If B_z increases from zero to its maximum value in a certain distance Δl, the component B_l perpendicular to the sector edge is $B_l = B_z z/\Delta l$. The force on the particle is $F_z = -(evB_l/c) \sin \gamma$ in the direction of the z-axis, and acts during a time $\Delta t \approx \Delta l/(v \cos \gamma)$. It follows that the vertical component p_z of the particle momentum varies by

$$\delta p_z = (eB_z z/c) \operatorname{tg} \gamma . \tag{2.54}$$

This means that when crossing the sector edge at an angle γ the particle changes the direction of its trajectory in the vertical plane by an angle

$$\chi_z = -Kz \operatorname{tg} \gamma . \tag{2.55}$$

The quantities χ_x and χ_z which characterize the effect of the edge on the radial and vertical motions differ only by their sign.

The excursions δx (or δz) reached by the particle after passing through the fringe field are small, of the second order with respect to χ: $\delta x \approx (Kx \operatorname{tg} \gamma)^2/2K$. So one can consider the effect of the fringing field at the edge as that of a thin magnetic lens which alters only the direction of flight of the particle. Thus, taking edge effects into account, a period can be written down as $M-K_1-O-K_2$, where M stands for the magnet sector, O for the straight section between sectors, and K_1, K_2 for the 'edge' magnetic lenses. Edge-focusing is described by matrices of the type $\begin{pmatrix} 1 & 0 \\ \pm Kl \operatorname{tg} \gamma & 1 \end{pmatrix}$. With the method of matrix multiplication applied above one can determine the closed orbit and obtain formulae for $\cos \mu$

$$\cos \mu_{x,z} \approx c - hs \pm s \operatorname{tg} \gamma/2\kappa . \tag{2.56}$$

Usually it is the 'gradient-focusing', given by the field dependence on radius, which plays the main role in accelerators, while edge-focusing introduces only small corrections. However, one large synchrophasotron [30] is in construction where the magnetic field is homogeneous in the sectors, as this is considered advantageous by the authors of the project. In this machine the stability of the motion in the vertical direction is realized only by edge-focusing. It should be noted that edge-focusing plays an important part in accelerators with azimuthally varying magnetic field, such as the ring-phasotron with spiral ridges (§ 3) and the isochronous cyclotron (§ 1).

3. STRONG-FOCUSING ACCELERATORS WITH FIXED MAGNETIC FIELD (RING-PHASOTRONS)

For azimuthally symmetrical accelerators with fixed field such as cyclotrons, one uses 'solid' (i.e. not annular) magnets with a field that is nearly homogeneous as a function of radius (see § 1). This leads to large weight and high power consumption and sets a limit to the energy one can reach (≈ 1 GeV for protons). These accelerators are not used in practice for electrons.

In recent years accelerators are being elaborated in which particle acceleration up to high energy can take place in a relatively narrow annular region of a fixed (time-independent) magnetic field. The magnetic guiding field has to vary considerably with radius in the limits of this annular region in accordance with the growth in energy. If the field were azimuthally symmetrical a strong instability would appear either in the radial or in the vertical direction, as one of the stability conditions of betatron oscillations ($0 < n < 1$) would not be satisfied.

In the article of reference [31], dating back to 1953, a method of avoiding this difficulty is indicated, making it possible to build a new type of accelerator with an annular magnet and a fixed magnetic field. The method essentially consists of introducing into the magnet, sectors with reversed magnetic field, producing strong focusing as the field and particle energy rise rapidly with radius. After this accelerator, which has been called the ring-phasotron (see also [32]), another type of accelerator was proposed which has the properties mentioned above, but with a more complicated, spiral field shape [32].

Thus, among strong-focusing fixed-field accelerators* one should distinguish between those of the radial-sector type and those of the spiral-sector type. In Soviet literature accelerators of both types are called ring-phasotrons.

The invariance of the magnetic field with time makes it possible in principle to raise the beam intensity substantially as compared with accelerators with time-varying magnetic fields. In these ring-phasotrons it is then possible to accelerate up to relativistic energies heavy as well as light particles (electrons).

* In foreign literature such accelerators are abbreviated FFAG accelerators (fixed-field alternating-gradient). Reports on the first operating accelerators of this kind can be found in [50].

In the ring-phasotron the cycle repetition rate of the high-frequency modulation can be chosen much higher than the magnetic-field repetition rate, which determines the beam intensity in synchrotrons and synchrophasotrons, where the magnetic guiding field varies with time.

With the induction (betatron) regime of acceleration in the ring-phasotron, particles can be captured by the acceleration mechanism during a considerable fraction of the cycle ($\frac{1}{5}$ to $\frac{1}{10}$) whereas in the usual betatron with time-varying magnetic field this fraction is a hundred or or a thousand times less (see ch. 4 § 8).

A radial-sector accelerator has N periodic elements, each of which consists of two radial sectors with opposite directions of field (fig. 75). In absolute value these fields increase either away from (direct version) or towards the centre (reverse version). The equilibrium orbits are wavy curves consisting of parts with positive and parts with negative curvature (see fig. 75). In these parts the field index n, taken along the normal to the orbit, has a large value, and opposite signs. Thus the particle experiences alternately focusing and defocusing in each direction (vertical and radial). As a result, with proper choice of the parameters one can have strong focusing in both directions.

The existence of sectors with reverse-field polarity has the consequence that the orbit length and the mean radius of the machine are larger than in accelerators without reversed-field sectors. If one wants to have betatron oscillation stability in both directions x and z, it is impossible to make the angular dimension of the sectors with reversed field very small. This leads to an increase of the circumference factor G (ratio of maximum radius to minimum radius of curvature) up to 5 to 6 and even more, which is the main disadvantage of radial-sector ring-phasotrons. However, the increase in beam intensity makes it possible to accept this to a certain extent. Also the maximum value of the fixed-magnetic field in the ring-phasotron can, in principle, be made considerably higher than in synchrophasotrons with time-varying magnetic fields, which in turn makes it possible to keep the dimensions of the machine smaller.

The shape of the magnetic field in a spiral accelerator can be regarded as a succession of hills and valleys uniformly distributed around the azimuth, having the form of spirals making a small angle with the circular co-ordinates (fig. 76). Along a radius the magnetic field exhibits

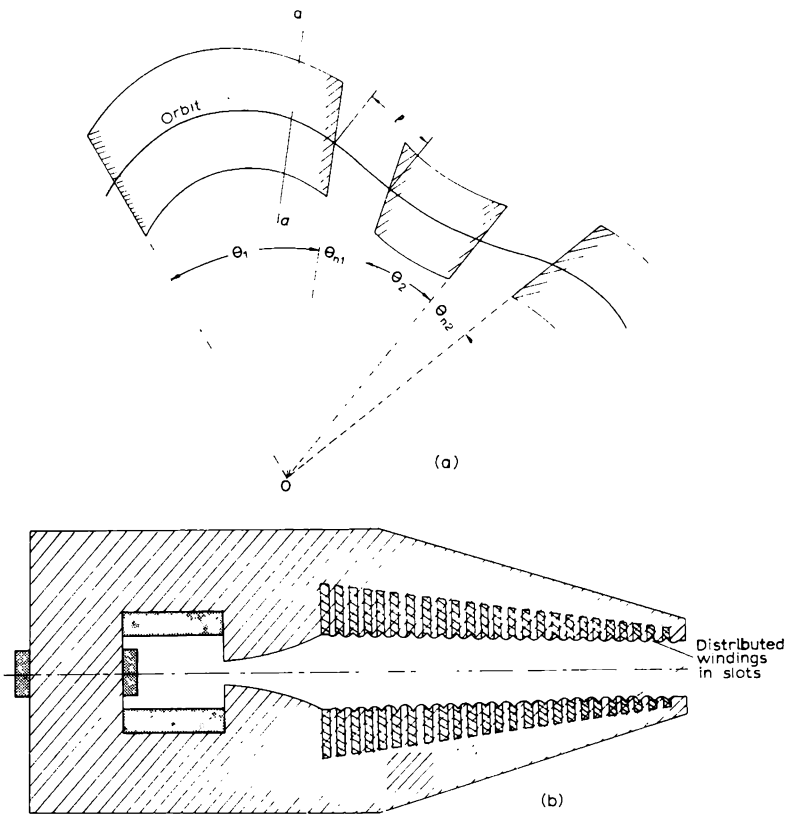

Fig. 75. (a) Schematic drawing of the elements of a period in a radial-sector accelerator: θ_1, θ_2 are the angular dimensions of the magnet sectors, θ_{n_1} and θ_{n_2} those of the straight sections; (b) Section along line aa, schematic.

variations corresponding to the hills and valleys, but on the average it increases as in the radial-sector case.

One can explain qualitatively the focusing in a spiral-ridge machine by the fact that in crossing the hills and valleys periodically, the particles move alternately in regions of positive and negative gradients. Thus, strong focusing is realized and it becomes possible to stabilize the z-motion by over-compensating the defocusing action caused by the radial increase of the mean magnetic field. As sectors with reversed field

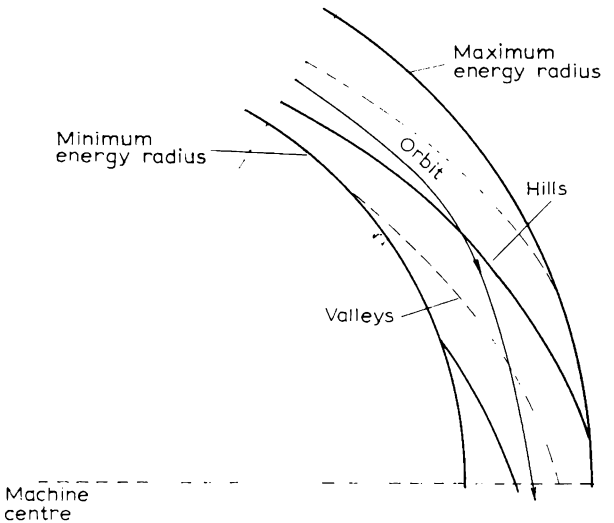

Fig. 76 Spiral-ridge accelerator (schematic).

are absent in the spiral-ridge accelerator, in contrast to the radial-sector machine, the circumference factor is much lower and can have values of 1.5 to 2. The disadvantage of the spiral-ridge accelerator is the intricate field shape, the realization of which presents technological difficulties and requires tight tolerances.

The symmetrical ring-phasotron (two-way FFAG accelerator). Let us examine briefly a variant of the ring-phasotron, which can be used to produce colliding beams of particles (see ch. 1 § 3). The structure which may be of either the radial-sector or the spiral-ridge type can be called the 'symmetrical' or 'two-way' phasotron, and was proposed in papers [33] and [34]. The simpler one, the symmetrical radial-sector machine, has sectors possessing identical geometrical and magnetic characteristics, except the sign of the field: $\theta_1 = \theta_2 = \theta_M$, $B_2(r) = -B_1(r) \propto r^{-n_0}$. The average field along a circle of radius r drawn from the centre O of the field has evidently the value zero. However, the closed orbits, around which betatron oscillations take place, are not circular but have the shape of a wave which passes through sectors with $B_1 > 0$ at larger radii

than through sectors with $B_2 < 0$, or vice-versa (see fig. 77). Thus, depending on the initial conditions of motion, the mean field along the orbit can be either positive or negative. Thus, it becomes possible to accelerate beams of identical particles (for instance electrons or protons) in opposite directions, clockwise and anti-clockwise. The orbits of the colliding beams have exactly the same geometrical form. They are only

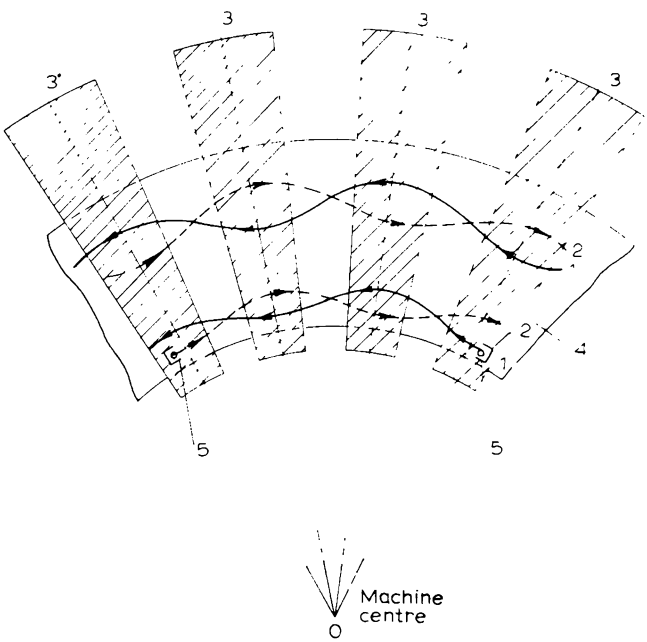

Fig. 77. Part of a symmetrical ring-phasotron. (1, 2) Orbits of particles travelling in opposite directions; (3) magnets; (4) vacuum chamber; (5) injector.

rotated by an angle π/N with respect to each other, so that there are $2N$ points where these orbits cross each other in each turn. With a suitable choice of parameters, the 'symmetrical' accelerator can be built so that radial and vertical betatron oscillations are stable. The characteristic values μ_x and μ_z are then the same in the two directions of revolution.

The identity of the sectors and of the characteristic numbers $\mu_{x,z}$ for the motion in the two directions, although desirable for simplicity and

symmetry, are not at all compulsory. In other words, a two-way accelerator has not necessarily to be symmetrical. Stable particle motion can be achieved in both directions with slightly different $\mu_{x,z}$-values.

Acceleration in the two-way machine does not require any more apparatus than in a one-way machine. In the resonance regime each of the colliding beams gets its energy from one and the same radiofrequency system. The equilibrium phases of the two beams will of course differ by π. It follows from the general theory of resonance self-focusing accelerators (see ch. 4) that the simultaneous acceleration of the two colliding beams cannot differ substantially from the acceleration of a beam in a normal synchrotron or phasotron.

One can store colliding beams accelerated in the phasotron regime during many acceleration cycles on an external orbit by means of a scheme described in Appendix B, § 4.

Scaling orbits. The question of scaling orbits occupies an important place in the theory of ring-phasotrons [32, 37]. As we saw in ch. 4 § 1, there exists in cyclic accelerators at every moment a family of closed periodic orbits depending on one parameter, the particle energy E (or the absolute value of particle momentum p). Let us write, as usual, $\mu_{x,z}$ for the number of radial and vertical betatron oscillations per turn around these orbits. As was shown in ch. 3, dangerous resonances occur at certain values $\nu_{x,z} = (\nu_{x,z})_{\text{res}}$, which can be integral, half-integral, etc. The betatron frequencies $\nu_{x,z}$ may vary for a number of reasons during acceleration. Thus, in accelerators with time-varying magnetic fields, the synchrotron oscillations and the energy and betatron oscillations they produce are accompanied by small periodic variations of μ_x and μ_z. In practice this requires only small corrections and does not present any danger of excitation of betatron oscillations.

In the ring-phasotron, however, where the magnetic field is constant in time, the energy spread of the beam will be much larger, as closed orbits exist simultaneously for all energies from the initial energy E_{min} to the final energy E_{max}. It is thus necessary to take special measures in order to limit changes in ν_x and ν_z in the course of acceleration. Let us require, for instance, that the shape of the magnetic field be such that the $\nu_{x,z}$-values should be constant for all closed orbits (all values of E and p)

$$\nu_x = \text{const.}, \qquad \nu_z = \text{const.}, \tag{3.1}$$

and different from the resonance values. We shall call orbits fulfilling condition (3.1) dynamically scaling. Let us determine the conditions the magnetic field must satisfy for the orbits to scale. We start from the equations

$$\text{(a)} \quad \frac{\partial}{\partial p}\left(\frac{K}{K_0}\right)\Big|_{\vartheta=\text{const.}} = 0, \qquad \text{(b)} \quad \frac{\partial n}{\partial p}\Big|_{\vartheta=\text{const.}} = 0, \tag{3.2}$$

which are the conditions for which the betatron oscillations are invariant (ch. 2, eqs. 2.13a and 2.13b) and therefore also the quantities $\nu_{x,z}$ when one passes from an orbit to a neighbouring one. Equation (3.2a) expresses the condition of geometrical similarity of the orbits and (3.2b) the condition of constancy of n at corresponding orbit points.

As one sees, besides the family of orbits depending on p, one can consider a family of curves $\vartheta = \text{const.}$ which are the loci of corresponding points. Let us call s the path along the curves $\vartheta = \text{const.}$ The geometrical similarity means that all the orbits can be obtained from one another simply by varying the scale; that is, for all the points on the curve, s is proportional to r

$$\left(\frac{\partial s}{\partial r}\right)_{\vartheta} = \sqrt{1 + r^2\left(\frac{\partial \theta}{\partial r}\right)^2_{\vartheta}} = \text{const.}, \tag{3.3}$$

r and θ being the usual polar co-ordinates in the plane $z = 0$. It then follows that

$$r\left(\frac{\partial \theta}{\partial r}\right)_{\vartheta} = h = \text{const.}, \tag{3.4}$$

where the parameter h should not depend on ϑ, as the field is unambiguous. We notice that the scaling condition, which involves keeping constant the angle between the orbit and the radius vector at corresponding points, leads exactly to the same requirement (3.4). This shows that the lines $\vartheta = \text{const.}$ are logarithmic spirals $\theta \propto \ln r$.

We introduce a 'geometrical' field index n_Γ, equal by definition to

$$n_\Gamma = -\frac{s}{B}\frac{\partial B}{\partial s} = -\frac{r}{B}\left(\frac{\partial B}{\partial r}\right)_{\vartheta}, \qquad B \equiv B_{z|_{z=0}}. \tag{3.5}$$

From the scaling property it follows that the quantities

$$\frac{K(p_1, \vartheta)}{K(p_2, \vartheta)} \quad \text{and} \quad \frac{p}{B}\left(\frac{\partial B}{\partial s}\right)_{\vartheta}\left(\frac{\partial s}{\partial p}\right)_{\vartheta}$$

do not depend on ϑ, while the value of Ks is independent of p when ϑ is constant. Making use of this, it is easy to show that n_Γ is independent of ϑ. On the other hand we can express the usual field index

$$n = -(KB)^{-1}(\partial B/\partial x)$$

in terms of n_Γ:

$$n = n_\Gamma \frac{1}{Ks}\frac{\partial s}{\partial x} - \frac{K'K_0}{K^2}\frac{\partial \vartheta}{\partial x}, \tag{3.6}$$

where x, as usual, is taken in the direction of the normal to the orbit (fig. 78). Taking into account that the dimensionless quantities in eq. (3.6), such as the values of n, are the same at corresponding points, we find that n_Γ is independent of p, that is

$$n_\Gamma = n_0 = \text{const.} \tag{3.7}$$

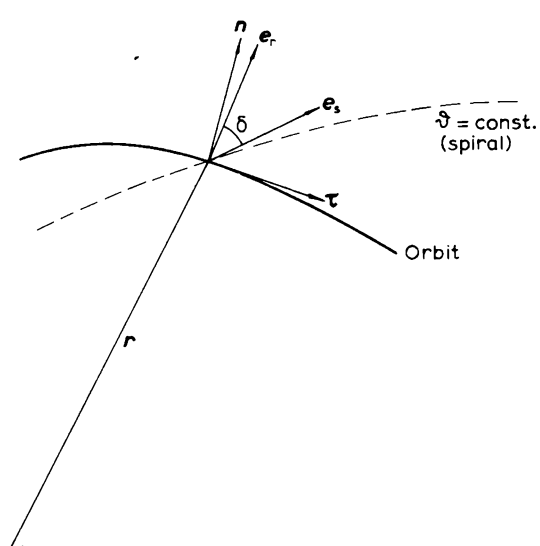

Fig. 78. Derivation of scaling condition.

From this we can conclude, according to (3.5), that a magnetic field satisfying the scaling condition (3.2) must have in general the form $B(\vartheta) = (s_i/s)^{n_0} F_1(\vartheta)$ or, in accordance with (3.3) and (3.4)

$$B(r, \theta) = B_i \left(\frac{r_i}{r} \right)^{n_0} F\left(\theta - h \ln \frac{r}{r_i} \right), \tag{3.8}$$

where

$$h = \operatorname{tg} \delta = \text{const.}, \tag{3.9}$$

δ being the angle between the field spiral and the radius vector, and B_i, r_i constants. The function $F(\vartheta)$ is an arbitrary periodic function with period $\vartheta_0 = \theta_0 = 2\pi/N$, where N is the number of periods. Let us note that the quantity $2\pi/(N \operatorname{tg} \delta)$ is the relative radial distance between two ridges (or valleys) of the field (3.8). The orbits are scaling in such a field (that is, satisfy condition 3.2a). However, the corresponding orbit points are not distributed on a radius vector ($\theta = \text{const.}$) as in the radial-sector case, but on logarithmic spirals:

$$\vartheta = \theta - h \ln \frac{r}{r_i} = \text{const.} \tag{3.10}$$

The orbits remain geometrically similar as they move away from the centre towards larger radii and at the same time they turn around the centre according to the law given in (3.10).

In radial-sector machines (where $h = \delta = 0$) corresponding orbit points ($\vartheta = \text{const.}$) lie on the same radial line, that is they have the same θ (in general ϑ is different from θ). The function $F(\theta)$ which characterizes the azimuthal field dependence is determined in practice by the azimuthal dimensions $\theta_1, \theta_2, \theta_{n_1}, \theta_{n_2}$ (see fig. 75), and also by the vertical magnet gap height, etc.

Let us now find the orbit function $\psi(\vartheta)$ and the orbit expansion coefficient α (momentum compaction factor) for fields which have the scaling property. By definition $\psi(\vartheta)$, the distance along the normal between orbits corresponding to momenta p and $p + \mathrm{d}p$, satisfies the equation $\mathrm{d}x = \psi(\vartheta) K_0^{-1} (\mathrm{d}p/p)$. Taking into account that $ps^{-1}(\partial s/\partial p)_\vartheta = \text{const.}$ we find that

$$\psi(\vartheta) \propto s \frac{\partial \boldsymbol{r}}{\partial s} \boldsymbol{n} \propto r \left(\frac{\partial \boldsymbol{r}}{\partial r} \right)_\vartheta \boldsymbol{n}, \tag{3.11}$$

$\boldsymbol{n}$ being the unit vector along the normal to the orbit at the point ϑ. Let us assume that $(\partial \boldsymbol{r}\ \partial r)_\vartheta = \mathbf{Q}\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is the unit vector in the direction of the radius and $\mathbf{Q}$ the linear operator for rotating by an angle $\delta = \text{arc tg } h$, which is constant because of the properties (3.9) and (3.10) of the logarithmic spiral. So we can write

$$\psi(\vartheta) = C(\mathbf{Q}\boldsymbol{r})\boldsymbol{n}\,, \tag{3.12}$$

where the constant coefficient C can be easily found from the relation $\alpha = \langle K K_0^{-1} \psi \rangle$. To integrate and take the average we shall use the Serret–Frenet formulae $K\boldsymbol{n}/K_0 = -\mathrm{d}\boldsymbol{\tau}/\mathrm{d}\vartheta$, $\mathrm{d}\boldsymbol{r}/\mathrm{d}\vartheta = \boldsymbol{\tau}/K_0$ (see ch. 2 § 1). We shall integrate by parts and remember that

$$\boldsymbol{\tau}(\mathbf{Q}\boldsymbol{\tau}) = \cos\delta = \frac{1}{(1+h)^{\frac{1}{2}}}\,. \tag{3.13}$$

Thus we arrive at the expression for $\psi(\vartheta)$

$$\psi(\vartheta) = \alpha K_0 (1+h^2)^{\frac{1}{2}} (\mathbf{Q}\boldsymbol{r})\boldsymbol{n}\,, \tag{3.14}$$

where h is the field spiral characteristic defined in (3.9) and (3.10). In particular we have for the radial-sector machine $\mathbf{Q} \equiv 1$, $h = 0$, and

$$\psi(\vartheta) = \alpha K_0 \boldsymbol{r}\boldsymbol{n}\,. \tag{3.15}$$

The momentum compaction factor α will be best derived from the relation between the orbit lengths

$$\Pi(p_2) = \Pi(p_1)\left(\frac{p_2}{p_1}\right)^{1/(1-n_0)} \tag{3.16}$$

for any momenta p_1 and p_2, because of the similarity of the orbits and the 'scaling' property of the field (3.8). From (3.16) we obtain for α a simple formula:

$$\alpha = \frac{1}{1-n_0}\,, \tag{3.17}$$

which is the same as that we had obtained in the case of an azimuthally symmetrical field with index n_0.

We notice that if we prescribe the orbit form $K = K(\vartheta)$, the equations (3.2), with n substituted from expressions (3.8), constitute a linear inhomogeneous system with periodic coefficients with respect to the

unknown ψn_Γ and ψ. According to general theorems [38] such a system can only have one periodic solution in the non-resonance case, and this solution has the same period ϑ_0 as the coefficients of the equation. It appears that the solution is given by $n = \text{const.}$ and by the expression of $\psi(\vartheta)$ we had derived in (3.14).

The scaling property affects not only the orbit form, it exerts a substantial influence on the whole dynamics of the accelerated particles [39]. In many questions scaling orbits have properties analogous to circular ones (see for instance 3.17). This analogy applies in particular to radiation effects. Let us derive as an example the expression for the decrement $\langle \zeta_x \rangle$ of radial betatron oscillations in a magnet system possessing scaling orbits. It is convenient to write (3.6) in the form

$$n = n_\Gamma \frac{\cos \delta}{K(\mathbf{Q}\boldsymbol{r})\boldsymbol{n}} + \frac{K_0 K'}{K^2} \frac{(\mathbf{Q}\boldsymbol{r})\boldsymbol{\tau}}{(\mathbf{Q}\boldsymbol{r})\boldsymbol{n}} . \tag{3.18}$$

Introducing (3.14) in (3.10) of ch. 5, and making use of (3.18), we obtain

$$\langle \zeta_x \rangle = \left\langle \tfrac{1}{2}\Gamma \left\{ 1 - K\alpha(1+h^2)^{\frac{1}{2}}(\mathbf{Q}\boldsymbol{r})\boldsymbol{n} + 2\alpha \left[n_\Gamma + \frac{K' K_0}{K} (1+h^2)^{\frac{1}{2}} (\mathbf{Q}\boldsymbol{r})\boldsymbol{\tau} \right] \right\} \right\rangle . \tag{3.19}$$

Taking into account that $\Gamma \propto K^2$, we integrate by parts and with the Serret–Frenet formulae obtain

$$(1+h^2)^{\frac{1}{2}} \left\langle 2\Gamma \frac{K' K_0}{K} (\mathbf{Q}\boldsymbol{r})\boldsymbol{\tau} \right\rangle = -(1+h^2)^{\frac{1}{2}} \langle \Gamma K_0 ((\mathbf{Q}\boldsymbol{r})\boldsymbol{\tau})' \rangle = -\langle \Gamma \rangle + \langle \Gamma K (\mathbf{Q}\boldsymbol{r})\boldsymbol{n} \rangle , \tag{3.20}$$

We introduce this into (3.19) and use (3.17) to arrive at

$$\langle \zeta_x \rangle = \frac{\langle \Gamma \rangle}{2} \frac{n_0}{1 - n_0} . \tag{3.21}$$

This expression is absolutely identical to (3.18) of ch. 5 for the motion in an azimuthally symmetrical magnetic field with $n \equiv n_0$. From (3.21) one can draw the same important conclusions for electron accelerators of the type considered here, as have already been discussed in ch. 5 §§ 3 and 4.

Determination of the orbits and oscillation frequencies. In the chapters devoted to the general theory of motion in magnet systems, we have used the natural system of co-ordinates (x, z, ϑ) related to the closed orbit. The results obtained may be, of course, applied to particle motion in ring-phasotrons. However, it is more convenient then to use co-ordinate systems not bound to the orbits, which must be determined themselves and may have a complicated form. Thus, it is adequate to make use of cylindrical co-ordinates (r, θ, z) which is also convenient when calculating arrangements for correcting the magnetic field. One can write the equations of the trajectories in cylindrical co-ordinates as follows

$$r''(r^2+z'^2)-r(r^2+2r'^2+z'^2)-rz'z'' = \frac{el^3}{pc}(rB_z - z'B_\theta), \quad (3.22a)$$

$$z''(r^2+r'^2)-r'z'(r+r'') = \frac{el^3}{pc}(r'B_\theta - rB_r), \quad (3.22b)$$

where primes indicate differentiation with respect to θ, and where

$$l = (r'^2+r^2+z'^2)^{\frac{1}{2}}. \quad (3.23)$$

Let us give an important property of the motion in the symmetry plane of the magnetic field $z=0$. We go over to a dimensionless radius $\rho = r/r_i$, where the radius of reference r_i is found from

$$p = \frac{e}{c} r_i B(r_i, \theta_i), \qquad B(r,\theta) \equiv B_z(r,\theta)|_{z=0}, \quad (3.24)$$

θ_i being some initial azimuth, and obtain the following equation

$$\rho\rho'' - 2\rho'^2 - \rho^2 = -(\rho^2+\rho'^2)^{\frac{1}{2}} \frac{B(\rho r_i, \theta)}{B(r_i, \theta_i)}. \quad (3.25)$$

From this we see that if the field $B(r,\theta)$ satisfies condition (3.8), the motion in the $z=0$ plane will take place along geometrically similar trajectories, provided the initial conditions are suitably chosen. These trajectories can, in principle, differ considerably from the closed orbits, so that the motion will be scaling, not only on a small scale (see 3.1) but also on a large scale [35].

To determine the orbits and betatron oscillation frequencies, let us write down the magnetic field $B(r, \theta)$ in the median plane given in (3.8) as a Fourier series

$$\frac{e}{pc} B(r, \theta) = \frac{\lambda(r)}{R} \sum_k b_k \exp \mathrm{i}Nk\,\theta\,, \tag{3.26}$$

where

$$\lambda(r) = \frac{eR}{pc} (\max B(r, \theta)) = \lambda_0 \left(\frac{R}{r}\right)^{n_0}, \tag{3.27}$$

R being a constant we shall define later. Because of scaling, the coefficients b_k, as well as n_0, λ_0 and h are independent of radius. Let us note that in the case of the two-way machine

$$b_0 = 0\,. \tag{3.28}$$

As will be shown in appendix G the properties of the motion of the particles in the system we are considering depend on two quantities which are given by the azimuthal shape of the field [40]

$$\Phi^2 = 4 \sum_1^\infty |b_k|^2\,, \tag{3.29}$$

$$S^2 = 2 \sum_1^\infty \frac{|b_k|^2}{k^2}\,. \tag{3.30}$$

The quantity Φ/b_0, sometimes called the field flutter or the field modulation coefficient, is a measure of the azimuthal field variation with respect to the average field. In spiral machines $\Phi/b_0 \approx 1$, in radial-sector ones $\Phi/b_0 \approx 6$. In the two-way machine $\Phi/b_0 \to \infty$ by definition.

To determine the parameters of the equilibrium orbit one has in general to integrate numerically eqs. (3.22) and look for periodic solutions. In practice, however, one uses approximate methods [25, 34, 41] based on the fact that the excursion of the radius vector of the orbit $r_0(\theta)$ away from some mean radius R is usually small

$$r(\theta) = R(1 + \xi(\theta))\,; \qquad \xi(\theta) = \sum_{k=-\infty}^{+\infty} \xi_k \mathrm{e}^{\mathrm{i}kN\theta}\,. \tag{3.31}$$

For the coefficients ξ_k we obtain to a first approximation from (3.22) (see also appendix G)

$$\underset{k\neq 0}{\xi_k} = -\frac{\lambda_0 b_k}{k^2 N^2}\,; \qquad \xi_0 = 0\,, \tag{3.32}$$

where

$$\lambda_0(R) = \frac{N^2}{2n_0 S^2}\left[-b_0 \pm \sqrt{b_0^2 - \frac{4n_0}{N^2}S^2}\right]. \tag{3.33}$$

As $\lambda_0(R) = \text{const.} \times R^{-n_0}$, relation (3.33) makes it possible to determine the mean orbit radius R, which does not depend in our approximation on the parameter h, that is on the field spiral.

From (3.32) and (3.33) we can deduce some general properties of the magnetic fields with the configuration we are considering. First, we notice the existence of two orbits corresponding to one energy*. For $n_0 < 0$, that is the direct version of the ring-phasotron, these orbits correspond to different signs of λ_0, so that the particles travel along them in opposite directions. In the symmetrical ring-phasotron (two-way machine) where $b_0 = 0$ the two orbits are absolutely identical

$$\lambda_0 = \pm \frac{N}{S(-n_0)^{\frac{1}{2}}}\,. \tag{3.34}$$

They are displaced azimuthally by an angle π/N with respect to each other. For $n_0 > 0$ (the inverse version of the ring-phasotron) the motion takes place in one and the same direction along both orbits. Let us note that the orbits only exist in this case if the average field is large enough: $b_0^2 > 4n_0 S^2/N^2$, while the symmetrical machine is not possible when $n_0 > 0$. Knowing the undulation $\xi(\theta)$ of the orbit, one can find the value of the circumference factor we have mentioned above

$$G = |K(\theta) r_0(\theta)|_{\max} = |\lambda_0 (1+\xi)^{1-n_0}|_{\max}\,. \tag{3.35}$$

The most favourable field from the point of view of circumference factor

* When the field has a large constant component $|b_0| < 1$, only one of these orbits is of practical interest, as the second one performs extremely large deviations from the mean circle.

is the azimuthally symmetrical field, where $G = 1$. In all other cases $G > 1$, as G increases with the field modulation (that is the flutter factor Φ/b_0).

To determine approximately the betatron oscillation frequencies $\nu_{x,z}$ we shall linearize eqs. (3.22) with respect to the orbit excursions $y = r - r_0$ and z, and apply the averaging method (see ch. 2 § 7). In doing so we shall assume that the representative point does not approach the upper limit of the betatron oscillation frequencies

$$\nu_{x,z}^2 \ll (\tfrac{1}{2}N)^2 . \tag{3.36}$$

For the unsymmetrical versions, that is for $b_0^2 \gg 4|n_0| S^2 N^2$ the betatron oscillation frequencies are given by the expression (see appendix G)

$$\nu_x^2 \cong 1 - n_0 + \frac{n_0^2 S^2}{N^2 b_0^2}, \tag{3.37}$$

$$\nu_z^2 \approx n_0 + \frac{n_0^2 S^2}{N^2 b_0^2} + \frac{2\Phi^2}{b_0^2}(h^2 + \tfrac{1}{2}) . \tag{3.38}$$

The terms $\pm n_0$ represent the focusing provided by the field gradient averaged over the azimuth, as in the case of weak focusing in an azimuthally symmetric field. The terms $n_0^2 S^2 / N^2 b_0^2$ represent the strong focusing produced by the gradient variation with azimuth arising from the undulation of the orbit. The term $\frac{1}{2}\Phi^2/b_0^2$ is the focusing given by the azimuthal variation of the field, similar to the Thomas focusing in isochronous cyclotrons. Finally, $\Phi^2 h^2/b_0^2$ is the specific strong focusing due to spiral field shape.

As one sees, the spiral shape of the field which is characterized by h has an effect mainly on z-focusing (compare with the isochronous cyclotron, § 1.2). However, it is exactly this kind of focusing which is required to compensate the defocusing caused by the sharp increase of the field with radius ($|n_0| \gg 1$). The absence of terms containing h in the expression for ν_x can be explained by the fact that for radial motion the effect of the spiral field variation is compensated by the azimuthal variation because of the undulation of the orbit. It should be noted that in spiral-ridge accelerators the strong-focusing and azimuthal-

variation effects can be relatively small. In this case, one may write approximately

$$\text{(a)} \quad v_x^2 \approx 1 - n_0 , \qquad \text{(b)} \quad v_z^2 \approx n_0 + \Phi^2 h^2 , \tag{3.39}$$

so that one must have $n_0 < 1$ to secure radial focusing.

For the symmetrical machine (see 3.28) we obtain instead of (3.37) and (3.38)

$$v_x^2 \approx -2n_0 , \tag{3.40a}$$

$$v_z^2 \approx -\frac{\Phi^2 N^2}{n_0 S^2}(1+2h^2) . \tag{3.40b}$$

In the expressions for ξ, G, v there enter, besides the obvious parameters N, n_0 and h, the numbers Φ, S, defined by the expansion coefficients of the field $B(r,\theta)$ in a Fourier series along the azimuth θ. To calculate these coefficients one should know the field form which depends, besides the parameters mentioned, on the ratio θ_1/θ_2 of the dimensions of positive and negative field sectors (for radial-sector machines), on the straight-section length, on the magnet gap height, etc. Clearly, sufficiently complete and accurate data on the field shape can be obtained from magnetic measurements on actual sectors. However, for rough estimates one can profitably make some simplifications, for instance, neglect stray fields, higher harmonics, etc.

It is often sufficiently accurate to assume that the azimuthal modulation follows the spiral law

$$b_k \equiv 0 \quad (k>1) , \qquad 2|b_1| = \Phi . \tag{3.41}$$

In the case of the usual spiral version this corresponds to the field

$$B(r,\theta) = B_i \left(\frac{r_i}{r}\right)^{n_0} \left[b_0 + \Phi \cos N\left(\theta - h \ln \frac{r}{r_i}\right)\right] . \tag{3.42}$$

The linearized betatron equations have then the form of the generalized Mathieu equations

$$u'' + (c_0 + c_1 \cos 2\tau + c_2 \cos 4\tau)u = 0 , \tag{3.43}$$

where the coefficients c_i depend only, when $|n_0| \gg 1$, on two combinations of the parameters n_0/N_2 and $\Phi h/(N|b_0|)$. The approximate expressions

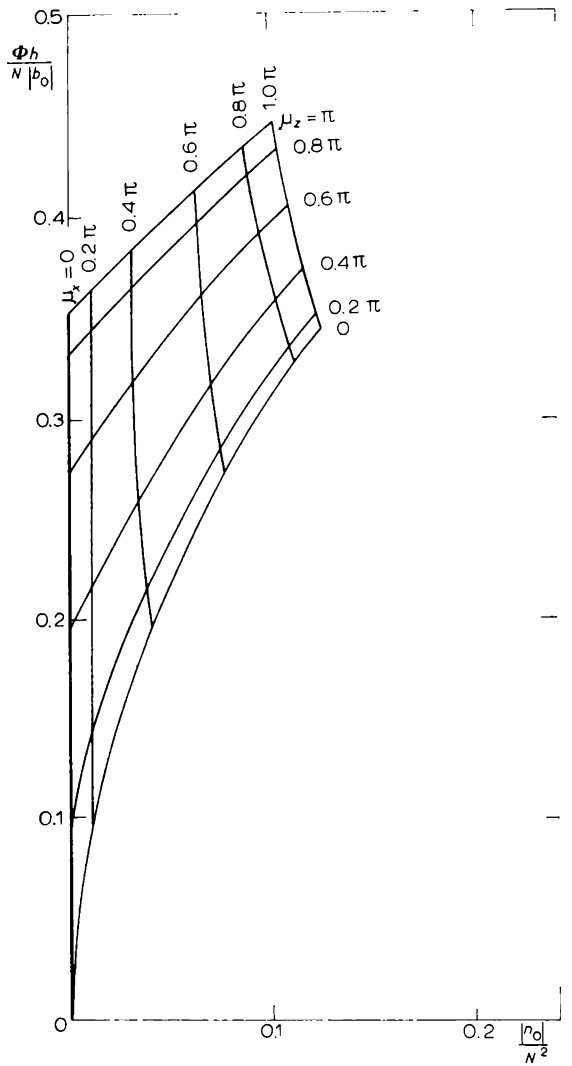

Fig. 79. Stability region of a spiral accelerator for $|n_0| \gg 1$ in the plane $\left(\frac{\Phi h}{N|b_0|}, \frac{|n_0|}{N^2}\right)$ according to [32].

for $\mu_{x,z} = 2\pi\nu_{x,z}$ N derived from (3.39) depend on the same two combinations. The lines $\mu_{x,z} = \text{const.}$ and the stability limits $|\cos \mu_{x,z}| < 1$, obtained by numerical methods, are given in fig. 79.

For a symmetrical radial-sector (two-way) machine we obtain with purely sinusoidal modulation of the field

$$B(r, \theta) = B_i \left(\frac{r_i}{r}\right)^{n_0} \cos N\theta \tag{3.44}$$

and with $|n_0| \gg 1$ the following relations:

$$\begin{aligned} \lambda_0 &\approx \mp N(-2/n_0)^{\frac{1}{2}} ; \\ \xi(\theta) &\approx \pm N^{-1}(-2/n_0)^{\frac{1}{2}} \cos N\theta , \end{aligned} \tag{3.45}$$

$$G \approx N(-2/n_0)^{\frac{1}{2}} + 2 , \tag{3.46}$$

$$\text{(a)} \quad \nu_x^2 \approx -2n_0 ; \qquad \text{(b)} \quad \nu_z^2 \approx -N^2/n_0 . \tag{3.47}$$

Thus, if the oscillations in one dimension, say x, correspond to the centre of the stability region ($\mu_x = \frac{1}{2}\pi$, $\nu_x = \frac{1}{4}N$), then $\nu_z \approx 4\sqrt{2}$, and vice versa. To the value $\nu_z = \frac{1}{4}N$ corresponds $|n_0| = 16$, and if one has at the same time $\nu_x \approx \frac{1}{4}$, then $N = 22$ to 23 and $G \approx 10$.

Some peculiarities of the acceleration regimes in ring-phasotrons (FFAG machines). It is interesting to compare the resonance acceleration regime in the ring-phasotron (FFAG) with the acceleration process in a synchrophasotron (synchrotron). In the latter there exists a well defined synchrotron orbit of length Π_0 which corresponds to the ideal law of variation of the accelerating frequency $\omega_s = \omega_s(t)$. If the radio-frequency $\omega_0(t)$ at a given instant differs somewhat from ω_s, that is if $\omega_0(t) = \omega_s(t) + \Delta\omega_0(t)$, the equilibrium orbit length Π_s will differ from Π_0 by a quantity $\Delta\Pi = \Pi_s - \Pi_0$. As there is only very little space available radially for the equilibrium orbits, one must put very tight tolerances on $\Delta\Pi$ and therefore also on $\Delta\omega_0$. We further note that in the case of the synchrophasotron the function $\mathcal{K}(E)$, which tends towards zero in the region $E = E_{cr}$, does not enter into the expression for $\cos \varphi_s \propto \dot{B}/V$.

The acceleration process in the ring-phasotron differs considerably from that in the synchrophasotron. All the closed orbits corresponding to the various energies are equally possible as at each instant t the

physically defined equilibrium orbit is the one for which the particle revolution frequency is equal to the frequency $\omega_0(t)$ of the accelerating field.

As the magnetic field is constant in time in the ring-phasotron, the revolution frequency is a function of the energy only, so that

$$\left(\frac{\partial\omega}{\partial E}\right)_B = \frac{d\omega}{dE} = -\mathscr{K}\frac{\omega}{E} = \frac{\omega}{E}\frac{1-\alpha\gamma^2}{\gamma^2-1}. \tag{3.48}$$

When applied to the ring-phasotron, this equation has a two-fold meaning. First, as in the synchrophasotron, it relates the quantities $d\omega$ and dE together, describing the deviations from the equilibrium values at a given instant. Secondly, and here there is a difference compared with the synchrophasotron, in the ring-phasotron the relation (3.48) links together the increase of frequency and the energy increase of the equilibrium particle in the time interval dt. In particular, if one puts $E = E_s$, one obtains from (3.48) an expression for the energy increase per turn of the equilibrium particle (ch. 4, eq. 3.17), which relates together the quantities V, $\cos\varphi_s$, ω_0 and $\mathscr{K}(E)$.

In the ring-phasotron the parameter α does not depend on energy because of the scaling. Thus, the integration of (3.48) yields a relation between energy and resolution frequency

$$\omega(\gamma) = \text{const.}\,(\gamma^2-1)^{\frac{1}{2}(1-\alpha)}/\gamma\,, \qquad \gamma = E/m_0c^2\,, \tag{3.49}$$

where the integration constant is determined by the absolute value of the magnetic field in the accelerator. Fig. 80 shows typical curves $\omega(\gamma)$ for various α. In particular, one sees that when a critical energy exists ($0<\alpha<1$), two energies correspond to each value of ω, one above and the other below critical energy.

Because of orbit scaling, the expressions for α and $\mathscr{K}(E)$ in the ring-phasotron are the same as for the azimuthally symmetrical phasotron (appendix C, § 1) with a field index $n = n_0$

$$\mathscr{K}(E) = 1 + \frac{n_0}{1-n_0}\frac{1}{\beta^2}. \tag{3.50}$$

However, in contrast to the ordinary phasotron, where $n \ll 1$, we have here $|n_0| \gg 1$. Thus, in the direct version ($n_0 < 0$) the quantity $\mathscr{K}(E)$ may

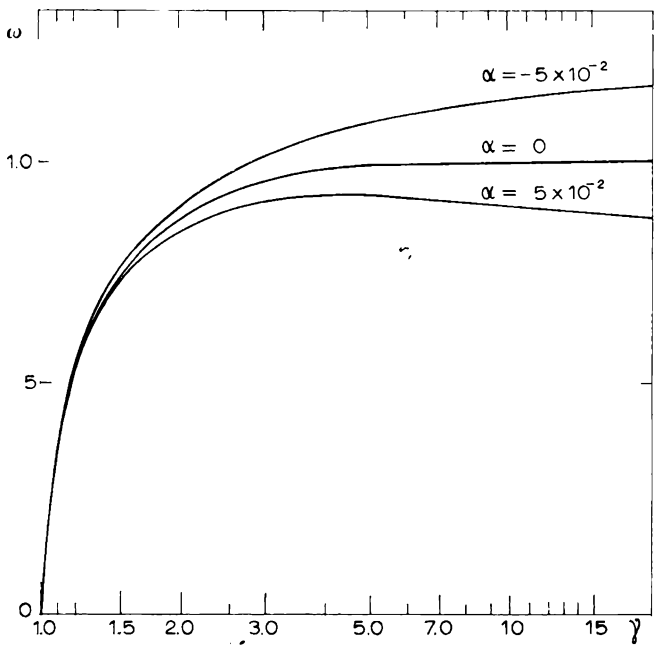

Fig. 80. The curves $\omega = \omega(\gamma)$ where $\gamma = E/m_0c^2$ in an accelerator with scaling orbits: $\alpha = 5 \times 10^{-2} > 0$, direct version; $\alpha = -5 \times 10^{-2} < 0$, inverse version.

become equal to zero, corresponding to the existence of a critical energy. In the inverse version ($n_0 > 0$) there is no critical energy.

In principle, an infinite number of functions $\omega = \omega(t)$ which fulfil the self-focusing condition $|\cos \varphi_s| < 1$ correspond to the uniquely defined quantity $\omega(\gamma)$. The limitations imposed on the radiofrequency $\omega_0 = \omega_0(t)$ by the self-focusing condition are not essential anywhere but in the region around E_{cr}. To find these limitations, one has to introduce into eq. (3.17) of ch. 4 the function $E_s(t)$ which can be obtained from the equation $\omega_0(t) = q\omega(\gamma)$, and from (3.49). If one does this, the dependence of the quantities E_s and $eV \cos \varphi_s$ may vary within broad limits according to the choice of the function $\omega_0(t)$.

If one sets the value of $V(t)$ one can derive the behaviour of the equilibrium phase φ_s in the course of the acceleration process. On the

contrary, if one decides the law of variation of $\varphi_s = \varphi_s(t)$, for instance $\varphi_s = \text{const.}$, one can always find various functions $\omega_0(t)$ and $V(t)$ which will realize this variation. Thus, the acceleration mechanism in the ring-phasotron offers many more possibilities in the choice of the parameters than in the synchrophasotron, with the result that the tolerances on $\Delta\omega_0$ may be substantially relaxed.

It is only in the region $E = E_{cr}$ that there are special requirements for the time dependence of ω_0 in ring-phasotrons [25, 42]. According to (3.49), one must lay down that for $t = t_{cr}$ (that is for $E_s = E_{cr}$) the function $\omega_0(t)$ passes through a maximum equal to $q\omega_{cr}$. This condition must be fulfilled with a high degree of precision. In fact, if $\omega_{0\,max} > q\omega_{cr}$, $\dot{\omega}$ does not go to zero when one arrives at the critical energy and self-focusing is lost. If $\omega_{max} < q\omega_{cr}$ the particles do not even reach critical energy and are slowed down when the frequency begins to decrease.

The passage through critical energy can be performed in the ring-phasotron by a phase jump of $2\varphi_s$ at the instant $t = t_{cr}$ as is done in synchrotrons. Several other methods have also been proposed for ring-phasotrons [42] and have already been checked in practice.

4. SPECIAL ACCELERATION MODES IN SYSTEMS WITH MAGNETIC FIELDS CONSTANT IN TIME

Systems with magnetic fields constant in time display two important features. First, particles with quite different energies, from injection energy to maximum energy, can travel at the same time. Secondly, if the vacuum is good enough the particles can live there for a long time, which opens the possibility of storing large circulating currents. Thanks to these properties, in fixed-field systems one can realize in addition to the usual mode of phasotron acceleration a whole series of other modes which are notable for their interesting features. In particular, these modes can open up new perspectives for machines like the ring-phasotron (see § 3).

4.1. Multiple acceleration in fixed field

When studying the properties of fixed-field systems, it is convenient to transform the phase equation into such a form that it describes the behaviour of particles at any energy, and not only in the neighbourhood

of E_s (see [42]). Going over in the general equations (ch. 4, eq. 3.18) to a new variable

$$u = \int \frac{dE}{\omega(E)}, \tag{4.1}$$

one may bring them into the canonical form for any energy

$$\begin{aligned} \frac{du}{dt} &= \frac{eV}{2\pi} \cos\varphi = \frac{\partial \mathscr{H}}{\partial \varphi}, \\ \frac{d\varphi}{dt} &= \omega_0 - q\omega(E) = \frac{\partial \mathscr{H}}{\partial u} \end{aligned} \tag{4.2}$$

with the Hamiltonian

$$\mathscr{H} = -\frac{eV}{2\pi} \sin\varphi - qE(u) + \omega_0 \cdot u\,. \tag{4.3}$$

It is not difficult to return from (4.2) to the previous representation of the phase motion. It is sufficient to carry out a single canonical transformation

$$u^* = u - u_s = \int_{E_s}^{E} \frac{dE}{\omega(E)}, \tag{4.4}$$

that is transform to a co-ordinate system related to the equilibrium energy. With the new variables the Hamiltonian has the form

$$\mathscr{H}^* = -\frac{eV}{2\pi} (\sin\varphi - \varphi \cos\varphi_s) - qE(u^*)\,. \tag{4.5}$$

If we expand $E(u^*)$ in powers of u^* and limit ourselves to quadratic terms, we obtain the Hamiltonian corresponding to the phase equation (ch. 4, eq. 3.23). If we continue the expansion up to terms cubic in u^* we obtain a much more general expression (ch. 4, eq. 7.16) which describes the particle behaviour in the critical region, etc.

Thus, the space of the variables (u, φ) appears to be the 'phase space', which allows us to derive a series of important conclusions, without going into a detailed study of the equations of motion [42 and 44].

First of all, on the basis of Liouville's theorem on the conservation of the phase volume occupied by the particles, one may state that if

the beam occupied a surface S on the phase plane at the beginning the energy spread in the beam cannot be less than

$$\Delta u = S/2\pi \,. \tag{4.6}$$

in whatever way the frequency and voltage of the accelerating field may vary with time*. In particular, if the beam covers uniformly the whole phase region from 0 to 2π at injection, the spread cannot be less after acceleration by the high-frequency field than

$$\Delta u_{\text{acc}} = \Delta u_{\text{init}} \tag{4.7}$$

or, for small Δu

$$\Delta E_{\text{acc}} = \Delta E_{\text{init}} \frac{\omega_{\text{acc}}}{\omega_{\text{init}}} \,. \tag{4.8}$$

This last relation plays an important part in the theory of storage rings.

On the basis of Liouville's theorem one can also draw a conclusion concerning the variation in energy of the particles lying outside the bucket, that is, not moving in synchronism with the high-frequency field. Let us consider particles with an energy definitely less than the equilibrium energy and uniformly spread in phase φ (fig. 81). Now let the

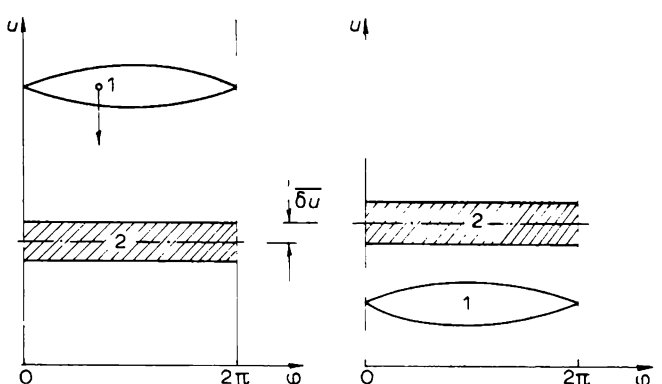

Fig. 81. Illustration of the phase displacement mechanism. 1, region occupied by the bucket; 2, region occupied by the beam; δu, mean displacement along co-ordinate u caused by the motion of the bucket.

* Here the conditions under which the equations (3.18) of ch. 4 were obtained should, of course, continue to apply.

frequency of the accelerating field vary slowly so that the equilibrium energy moves towards lower energies. After the region of closed phase trajectories has 'passed through' the beam, the mean energy of the particles is found to have increased, so that the bucket will have 'displaced' them so to speak towards higher energies. Proceeding from Liouville's theorem, one can expect that the displacement measured in terms of the co-ordinate u will be equal to $S/2\pi$, where S is the surface enclosed by the bucket. This displacement takes place mainly at the moment when the frequency has the value $q\omega(E)$ where the particle is excited in resonance for a short time. One can show that after each such traversal of a resonance, the particle energy changes by

$$\delta u(\varphi) = \frac{1}{\omega}\left[\frac{eVE}{2\pi q|\mathcal{K}|}\right]^{\frac{1}{2}} \int_{\mp\infty}^{\varphi} [\sin\xi - \sin\varphi - (\xi-\varphi)\cos\varphi_s]^{-\frac{1}{2}} \cos\xi \,\mathrm{d}\xi\,, \tag{4.9}$$

where all the quantities on the right-hand side should be taken at the moment when $u = u_s$ (that is $u^* = 0$). This is illustrated in fig. 82 where

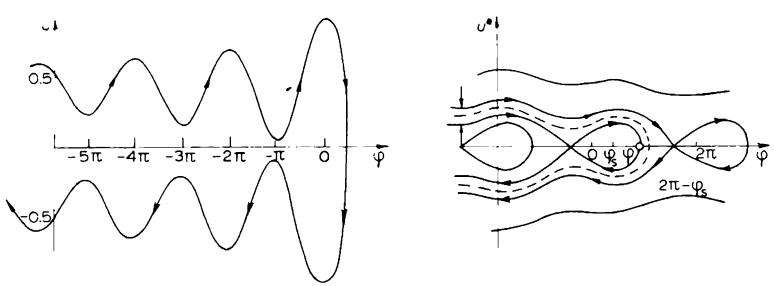

Fig. 82. Phase trajectories of particles lying in the unstable region. Displacement of the particles with respect to co-ordinate u as a function of phase φ.

the phase trajectory is shown of a particle which lies in the unstable region and passes at the side of the bucket. The upper or lower sign in (4.9) is to be taken depending on the inequality $\cos\varphi_s \gtrless 0$, that is, on the side from which the stable region approaches the particle in phase space.

The function $\delta u(\varphi)$ is represented in fig. 83 from which one sees that most of the particles are displaced in the direction opposite to the motion of the bucket. Only those particles whose phase trajectories pass near

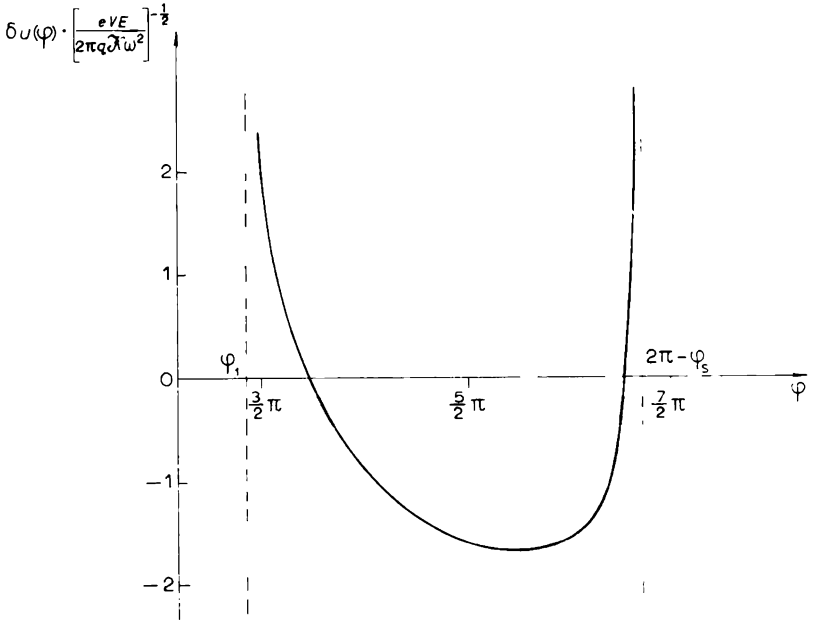

Fig. 83

the saddle-point of the bucket remain in its vicinity and are carried away by the bucket in the direction of its motion. One can assume that the phase φ entering in (4.9) is a random quantity whose distribution has the form [43, 44]

$$W(\varphi) = \frac{\cos\varphi_s - \cos\varphi}{2\pi\cos\varphi_s}, \tag{4.10}$$

where φ lies always between $2\pi-\varphi_s$ and φ_1. With (4.10) one can calculate a mean value $\overline{\delta u(\varphi)}$ which is equal to

$$\overline{\delta u(\varphi)} = \int \delta u(\varphi) W(\varphi) \mathrm{d}\varphi =$$

$$= \mp \frac{1}{\pi}\left[\frac{eVE}{\pi\omega^2 q|\mathcal{H}|}\right]^{\frac{1}{2}} \int_{-\varphi_s}^{\varphi_1} [\sin\varphi - \sin\varphi_s - (\varphi+\varphi_s)\cos\varphi_s]^{\frac{1}{2}} \mathrm{d}\varphi =$$

$$= \mp S/2\pi, \tag{4.11}$$

S being the surface of the stable region in phase space. Thus, during a whole acceleration cycle, particles lying in an unstable region of phase space are actually displaced in energy in a direction opposite to that of the bucket motion. The energy shift is equal to the surface enclosed by the bucket divided by 2π. This effect is called 'phase displacement'* and opens the way to a new method of acceleration: if one modulates the radiofrequency in the direction opposite to that which would have been necessary for self-focusing acceleration, the particles lying in the unstable phase-space region will increase their energy on the average at each modulation cycle as they are displaced by the successive buckets (see fig. 81). The basic disadvantage of this method is that the total acceleration time is much longer for these particles than with self-focusing, so that they remain a much longer time exposed to the effects of perturbations.

Another interesting acceleration method in fixed fields is multi-stage acceleration with variable harmonic number. Let us assume, for instance, that one has self-focusing acceleration with a harmonic number q, a frequency swing from $\omega_{0\,\mathrm{init}}$ to $\omega_{0\,\mathrm{final}} < \omega_{0\,\mathrm{init}}$, and an energy increase from $E_{s\,\mathrm{init}}$ to $E_{s\,\mathrm{final}} > E_{s\,\mathrm{init}}$. If the harmonic number is large enough, one can make the required frequency bandwidth smaller by varying the frequency only from $\omega_{0\,\mathrm{init}}$ to a certain frequency ω_{01} ($\omega_{0\,\mathrm{init}} < \omega_{01} < \omega_{0\,\mathrm{final}}$). For this the condition must be fulfilled

$$\frac{q}{q_1}\,\omega_{0\,\mathrm{final}} = \omega_{01} = \frac{q_1}{q}\,\omega_{0\,\mathrm{init}}\,, \tag{4.12}$$

where q_1 is an integral number ($q_1 < q$). Evidently this can be realized if the value of q is large enough, of course, only to a certain degree of accuracy. Under these circumstances two buckets exist simultaneously in phase space**, and particles with an energy $E \approx E_{s\,\mathrm{init}}$ as well as an energy $E \approx E_{s1}$ can be captured at the same time into the self-focusing acceleration mode. The former are accelerated in each working cycle with harmonic number q up to the energy E_{s1}, then they are captured at the beginning of the next cycle by the mode with harmonic number q_1 and accelerated up to the maximum energy. More complicated acceler-

* In the foreign literature: phase displacement mechanism.

** We assume for simplicity that there is no critical energy.

ation mechanisms also exist, in which the modes described are combined with usual phasotron or betatron acceleration. Let us stress that in all cases considered, the acceleration mode is derived more or less closely from the self-focusing one and requires that both amplitude and frequency of the accelerating voltage vary in a predetermined manner. Quite an interesting exception to this rule is the so-called stochastic acceleration to which we shall come back in a special section.

4.2. Storage devices

Thanks to the fact that in fixed-field machines the particles can travel for a long time at constant energy if the vacuum is sufficiently good, it becomes feasible to store large currents of accelerated particles exceeding by many times the currents circulating in usual accelerators. This is the principle of the storage devices intended for colliding-beam experiments (see ch. 1 § 3).

The importance of storing large currents for such experiments is obvious, as each beam acts as a target of extremely small density for the other one. One easily finds that the number of interactions per unit time $N(t)$ is proportional to the square of the total number of particles N_0

$$N(t) = \frac{2v\sigma N_0^2 w}{W_0^2}, \tag{4.13}$$

where W_0 is the total volume occupied by the beam, w the volume where interactions take place and σ the cross-section of the process one wants to investigate. This is why the number of interactions is vanishingly small for small beam densities, and begins to be acceptable only for large values of N_0. As an example, we can mention that for the study of large-angle scattering of 500 MeV electrons a current of 1 A is required in each beam. Apart from this aspect of the question, one should remember that the number of background counts in experiments of such a type increases only as N_0, so that a large intensity is also desirable for increasing the signal-to-noise ratio.

One can divide storage devices in two groups: those in which acceleration and storage take place in the same magnet, and those which are filled from outside. Fixed-field accelerators (in particular strong-

focusing ring-phasotrons) fall into the first category. After each acceleration cycle the frequency of the accelerating voltage is made to return rapidly to the initial value, so that the particles remain on the maximum-energy orbit. After this a new batch of particles is accelerated and so on.

The subsequent acceleration cycles have, of course, an effect on the already stacked beam, especially at the end of the cycle, where $\omega_0(t)$ becomes equal to a multiple of the revolution frequency $\omega(E)$ of the stacked beam. It is not difficult to estimate this effect with the help of Liouville's theorem. As the density in phase space remains constant the stacked beam is displaced in energy after each acceleration cycle, as if it made room for the following bunches of particles. In the end the energy spread of the stacked beam increases gradually and cannot be less than

$$\Delta E_{\text{stacked}} = \Delta E_{\text{init}} \frac{\omega_{\text{acc}}}{\omega_{\text{init}}} k , \tag{4.14}$$

k being the number of stacks and ΔE_{init} the energy spread in the beam at injection. Such an increase of the energy spread is an unavoidable consequence of the way storage devices work, unless a dissipative process produces some damping, as does radiation (see ch. 5).

Storage in an accelerator like the symmetrical ring-phasotron is especially interesting, as two beams travelling in opposite directions can be simultaneously accelerated and stored in this machine (see § 3), which can be used for colliding-beam experiments [33, 35, 36].

In storage devices with external filling [45] particle acceleration takes place in a separate machine. It is convenient to take linear rather than cyclic accelerators for this purpose, as there are then no extraction difficulties. After leaving the accelerator, the particles are introduced into the fixed-field storage device especially designed for colliding-beam experiments. This may consist for instance of two rings with a section in common. One injects the accelerated particles alternatively in each of the two rings in such a manner that the two beams collide in the device (fig. 84).

The main difficulty encountered in such a storage scheme is the injection of the high-energy particles into the fixed-magnetic field. One can do this with single turn injection (see ch. 1 § 6), whereby the magnetic field in some section is perturbed during a time comparable with the

period of revolution in the storage ring. Electron storage devices are now under study where the radiation emitted by the electrons in the magnetic field plays an important part in the injection and storage processes (see ch. 5). This radiation makes the orbit shrink away from the injector and provides damping of betatron and synchrotron oscillations.

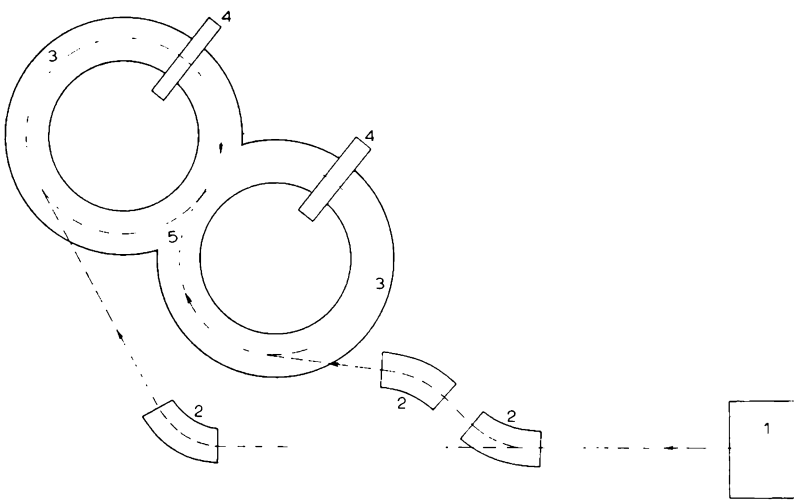

Fig. 84. Storage device with external filling, schematic [45]. 1, linear accelerator; 2, bending magnets; 3, particle orbits; 4, cavities to compensate radiation losses; 5, collision region.

Storage efficiency depends mainly on the mean lifetime of the particles at the storage energy. This lifetime is determined by the vacuum which one can attain in storage rings. Estimates show that in the examples given above the vacuum should be at least of the order of 10^{-8} to 10^{-9} mm Hg.

One sees that the practical realization of storage is a very difficult task and that the theoretical considerations studied so far serve only as a general guide.

4.3. *Stochastic acceleration*

In cyclic accelerators with fixed fields, as well as resonance and induction acceleration, another scheme called stochastic acceleration is also possible. It was first proposed in the paper [46]. Its main feature is the use of a random acceleration mechanism.

To make the principle clear, let us consider the following greatly simplified explanation. Let a particle with an initial energy E_{init} rotate in the magnetic field and pass many times through an accelerating gap to which a voltage varying in time is applied. For simplicity we shall consider that the voltage may take the two values $+V$ and $-V$ with equal probability, so that the particle can either receive or lose energy.

The probability that the particle will traverse the gap k times in succession when the field is accelerating and gain an energy keV is equal to 2^{-k}. This is, of course, vanishingly small for large k. However, we should remember that a particle can travel as long as one wishes under ideal conditions in a fixed-magnetic field if its energy lies within defined limits, say $E_{\text{init}} \leqslant E < E_{\max}$. Let us ask what fraction of the injected particles reaches the maximum energy in any number of revolutions, having been successively accelerated and decelerated in a random manner. In so doing the particle will accomplish a motion comparable to that of a Brown particle which would move in a liquid from one side of a vessel to the other.

Let us now estimate the probability that a particle reaches the energy $E_{\max} = keV$, k being very large compared with 1, by all possible paths without the particle energy falling below E_{init}. For simplicity let us consider the series of energy values $E_k = E_{\text{init}} + keV$, where k takes values corresponding to a geometrical series $k = 1, 2, 4, 8, \ldots$ The probability that the particle reaches the energy E_1 without being lost after the first revolution is obviously $\frac{1}{2}$. Particles with energy E_1 have an equal probability either to see their energy decreased by eV and to be lost, or to see it increased by eV and to reach energy E_2. Thus, the probability that the particle reaches the energy E_2 is 2^{-2}. In a similar manner a particle with energy E_2 either comes down to the initial energy $E_{\text{init}} = E_2 - 2eV$ and is lost after a sufficiently large number of turns, or arrives, even if only once, with a probability $\frac{1}{2}$, at the energy $E_4 = E_2 + 2eV$. In general, one can write that the fraction of the particles leaving the source

which reaches the energy E_k is

$$\mathscr{I}_k = \frac{1}{2k} = \frac{1}{2}\left[\frac{eV}{E_k - E_{\text{init}}}\right]. \tag{4.15}$$

One can generalize this result for any value $k \neq 2^m$ by methods from the theory of random processes. This generalization does not change the dependence of the full probability $\mathscr{I}_k$ on k given in (4.15) very much. The fact that the particles are not accelerated any more for $E > E_{\max}$ as well as for $E < E_{\text{init}}$ has no real bearing on this result.

One can satisfy oneself that the probability (4.15) corresponds to significant beam intensities at the output of the accelerator considered (the stochatron) if one remembers that the accelerated particles are obtained continuously, and not in bursts as in the synchrocyclotron, synchrotron, etc.

The mean number of times the particle has to cross the accelerating gap to reach an energy E_k is, of course, larger than k and of the order of k^2. It so happens that the interval of time needed for this is not so large that scattering by residual gas and other phenomena can substantially perturb the acceleration.

Let us now consider the laws of stochastic acceleration under more realistic conditions [44, 47]. First, the relatively small increase in energy per turn leads us to consider stochastic acceleration as a diffusion process where the energy or the variable u plays the role of a co-ordinate. Thus, it is necessary if one wants to make numerical estimates, to compute the 'diffusion coefficient' $D = \frac{1}{2}(\overline{\mathrm{d}u^2/\mathrm{d}t})$, where the bar indicates averaging over all particles present in the chamber. We shall consider also that one applies a voltage $V(t)$ which is a random function of time, that is noise, with a spectral density $\Phi(\omega)$ to the accelerating gap of the stochatron. One can then write the accelerating field along the orbit corresponding to energy E as the following function of generalized azimuth and time.

$$\mathscr{E}_\tau(\vartheta, t) = \frac{V(t)}{\Pi(E)}\,\mathrm{g}(\vartheta), \tag{4.16}$$

where the mean value of $\mathrm{g}(\vartheta)$ along the azimuth must be $\langle \mathrm{g} \rangle = 1$. We shall not impose any further limitations on the form of function $\mathrm{g}(\vartheta)$.

The equations of motion in terms of the variables u, ϑ have the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{ve\mathscr{E}_\tau(\vartheta, t)}{\omega(E)} = \frac{eV(t)}{2\pi}\,\mathrm{g}(\vartheta)\,, \qquad \frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \omega(E)\,. \tag{4.17}$$

By analogy with ch. 4 § 6, where we had examined the effect of noise, we obtain

$$\overline{\mathrm{d}u/\mathrm{d}t} = 0\,, \tag{4.18}$$

$$D = \frac{e^2}{4\pi^2}\int_0^t V(t')\,V(t)\,\mathrm{g}\left(\int_{t'}^{t}\omega\,\mathrm{d}t\right)\mathrm{g}\left(\int_{t'}^{t}\omega\,\mathrm{d}t\right)\mathrm{d}t'\,, \tag{4.18}$$

where the wavy line indicates time averaging*. Let us expand $\mathrm{g}(\vartheta)$ as a Fourier series

$$\mathrm{g} = \sum_k \mathrm{g}_k \exp \mathrm{i}k\vartheta = \sum_k \mathrm{g}_k \exp \mathrm{i}k\int_t^t \omega\,\mathrm{d}t\,. \tag{4.19}$$

Considering that the energy E and therefore ω change little in the interval of correlation of the process $V(t)$, we obtain

$$D = \frac{e^2}{4\pi}\sum_{k,l}\int_0^t V(t-\tau)\,V(t)\exp \mathrm{i}\left[l\int_t^t \omega\,\mathrm{d}t - k\omega\tau\right]\mathrm{g}_k\,\mathrm{g}_{l-k}\,\mathrm{d}\tau =$$

$$= \frac{e^2}{4\pi}\sum_k |\mathrm{g}_k|^2 \int_0^t \Psi(\tau)\,\mathrm{e}^{-\mathrm{i}k\omega\tau}\,\mathrm{d}\tau\,, \tag{4.20}$$

where $\Psi(\tau)$ is the correlation function of the process $V(t)$; terms with $l \neq 0$ give zero after being averaged in time. Making use of the relation between the correlation function and spectral density (see ch. 4, eq. 6.15) we obtain finally

$$D = \frac{e^2}{4\pi}\sum_{k=1}^{\infty} |\mathrm{g}_k|^2\,\Phi(k\omega)\,. \tag{4.21}$$

* We consider only stationary random processes displaying the ergodic property (see ch. 4 § 6).

In particular, when the accelerating field is concentrated inside a narrow gap

$$g(\vartheta) = 2\pi\delta(\vartheta), \qquad g_k = 1 \tag{4.22}$$

and

$$D = \frac{e^2}{4\pi} \sum_{k=1}^{\infty} \Phi(k\omega). \tag{4.23}$$

If there is no correlation in the course of one revolution, one can consider the spectrum as constant over an interval equal to ω, so that (4.23) yields

$$D = \frac{e^2}{4\pi} \int_0^{\infty} \Phi(k\omega)\,\mathrm{d}k = \frac{e^2 \tilde{V}^2}{4\pi\omega}. \tag{4.24}$$

It is just this case that we considered above, as we assumed that the particle receives at each revolution statistically independent energy increments, equal to $\pm V$. Let us mention that the correlation distance characterizes the spectral bandwidth $\Delta\omega$ of the noise: $\tau_{\mathrm{corr}} \approx 2\pi\,\Delta\omega$ so that physically one has no more correlation within one revolution when the width of the spectrum becomes much greater than the revolution frequency.

Formulae (4.20) and (4.23) have a simple physical meaning: the particle chooses, so to speak, the harmonics which are a multiple of its revolution frequency and act in an incoherent manner.

We have found already that, if $V(t)$ is sufficiently small, the acceleration process can be considered as a diffusion along the axis u with a coefficient $D(u)$. For stationary operation of the stochatron, we get the distribution $U(u)$ of the particles from the Einstein-Fokker equation

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} DU = \frac{U}{t_0(u)} - F_{\mathrm{inj}}(u). \tag{4.25}$$

On the right-hand side we find the terms describing the arrival of new particles from the injector F_{inj} and the losses due to the mean lifetime $t_0(u)$. The boundary conditions for $U(u)$ are

$$U(0) = U(u_{\max}) = 0, \tag{4.26}$$

where $u = 0$ and $u = u_{\max}$ correspond to the minimum and maximum

energies between which the particles can move in the magnet without hitting the chamber walls. In stationary conditions of operation a dynamic equilibrium is reached between the arrival of new particles from the injector at $u = u_{inj}$ and the diffusion currents lost on the 'absorbing walls' at $u = u_{max}$ and $u = 0$. The useful current of the stochatron is equal to the diffusion current at $u = u_{max}$

$$\mathcal{I}_{output} = -\frac{d}{du} DU\Big|_{u_{max}} \tag{4.27}$$

and must be found from the solution of the diffusion equation (4.25), taking into account the actual dependence of D and t_0 on u.

It is easy to calculate the mean time $\bar{t}$ which the particles take to reach maximum energy in the stochatron. As $\overline{du^2/dt} = 2D(u)$, we obtain

$$\bar{t} \approx \int_{u_{inj}^2}^{u_{max}^2} \frac{du^2}{2D(u)}, \tag{4.28}$$

or, putting $\omega \approx$ const. to obtain an estimate,

$$\bar{t} \approx \frac{E_{max}^2 2\pi}{\omega e^2 \tilde{V}^2}. \tag{4.29}$$

In other words, the particle accomplishes on the average $E_{max}^2/e^2\tilde{V}^2$ revolutions, which is about E/eV times more than in the phasotron regime. In order to prevent the relatively long acceleration time causing particle losses, stricter requirements should be laid down for the vacuum in a stochatron.

As the theory of the stochatron, and in particular questions concerning the disadvantages of the system, are not yet known well enough, we shall not give any numerical estimates here. We shall only mention that in any case the beam intensity at the output depends on the form of the function $D(u)$ and, therefore, on the choice of the spectrum of the accelerating noise. A general rule is to make the spectral density as large as possible at those frequencies which correspond to the smallest lifetime $t_0(u)$ in order to bring the particles away from the dangerous regions (for instance, the injector) as quickly as possible. By optimizing the spectrum

in this way, one may hope to overcome the defects of the stochastic acceleration scheme, which are the high radiofrequency power required and the difficulties at injection.

We should mention that the stochastic regime can also be useful in combination with other acceleration methods, for instance, to facilitate injection into the synchrocyclotron, to pass through transition energy, etc. Some of these applications are briefly examined in references [48, 49], where the first working model of the stochatron is also described. The stochastic regime can also be applied to store particles on a given orbit by appropriate choice of the spectrum $\Phi(\omega)$ as shown in [47].

APPENDICES

APPENDIX A. EQUATIONS OF MOTION AROUND A THREE-DIMENSIONAL ORBIT

If the twisting of the orbit is different from zero, the co-ordinate vectors will be expressed by $\boldsymbol{\tau}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ as follows

$$\boldsymbol{a}_1 = \boldsymbol{n}\ ; \qquad \boldsymbol{a}_2 = \frac{1}{K_0}\left[\tau(1+Kx)+\kappa(z\boldsymbol{n}-x\boldsymbol{b})\right]\ ; \qquad \boldsymbol{a}_3 = \boldsymbol{b}\ , \tag{A.1}$$

whereas the determinant g is given as before by the formula

$$g = \left(\frac{1+Kx}{K_0}\right)^2 . \tag{A.2}$$

The covariant and contravariant components of the magnetic field in the natural co-ordinate system have the form

$$B^1 = B_x - \frac{\kappa z}{1+Kx}B_\vartheta\ ; \qquad B^2 = \frac{K_0 B_\vartheta}{1+Kx}\ ; \qquad B^3 = B_z + \frac{\kappa x B_\vartheta}{1+Kx}\ ;$$

$$B_1 = B_x\ ; \qquad B_2 = \frac{1}{K_0}\left[B_\vartheta(1+Kx)+B_x\kappa z - B_z\kappa x\right]\ ; \quad B_3 = B_z\ . \tag{A.3}$$

Introducing these expressions in Maxwell's equations we obtain the following relations between the components of the field

$$K_0\frac{\partial B_\vartheta}{\partial\vartheta} + \frac{\partial}{\partial x}\left[B_x(1+Kx)-\kappa z B_\vartheta\right] + \frac{\partial}{\partial z}\left[B_z(1+Kx)+\kappa x B_\vartheta\right] = 0\ ,$$

$$\frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z}\ ; \qquad K_0\frac{\partial B_z}{\partial\vartheta} = \frac{\partial}{\partial z}\left[B_\vartheta(1+Kx)+B_x\kappa z - B_z\kappa x\right]\ ; \tag{A.4}$$

$$K_0\frac{\partial B_x}{\partial\vartheta} = \frac{\partial}{\partial x}\left[B_\vartheta(1+Kx)+B_x\kappa z - B_z\kappa x\right] .$$

If we take the projection of eq. (1.7) of ch. 2 on the directions $\boldsymbol{n}$ and $\boldsymbol{b}$, we obtain the following equations of motion

$$\left[\frac{d\vartheta}{dt}\left(x' + \frac{\kappa z}{K_0}\right)\right]' =$$

$$= \left\{\frac{1+Kx}{K_0}\left(\frac{ecB_z}{E} + \frac{K}{K_0}\frac{d\vartheta}{dt}\right) - \left(\frac{ecB_\vartheta}{E} + \frac{\kappa}{K_0}\frac{d\vartheta}{dt}\right)\left(z' - \frac{\kappa x}{K_0}\right)\right\}; \quad \text{(A.5)}$$

$$\left[\frac{d\vartheta}{dt}\left(z' - \frac{\kappa x}{K_0}\right)\right]' =$$

$$= \left\{\left(\frac{ecB_\vartheta}{E} + \frac{\kappa}{K_0}\frac{d\vartheta}{dt}\right)\left(x' + \frac{\kappa}{K_0}z\right) - \frac{ec}{E}B_x\frac{(1+Kx)}{K_0}\right\}, \quad \text{(A.6)}$$

where

$$\frac{d\vartheta}{dt} = \frac{K_0 pc^2}{E}\left[(1+Kx)^2 + (x'K_0 + \kappa z)^2 + (z'K_0 - \kappa x)^2\right]^{-\frac{1}{2}}. \quad \text{(A.7)}$$

One can easily satisfy oneself that the linearization of eqs. (A.5) and (A.6) leads to the system (2.14) of chapter 2.

APPENDIX B. THE LIAPUNOV–POINCARÉ THEOREM

Let us consider two possible solutions $\boldsymbol{X}_1(\vartheta)$ and $\boldsymbol{X}_2(\vartheta)$ of the system (5.10) in chapter 2. One can verify directly that there exists an integral of motion

$$x_1(\vartheta)p_{x_2}(\vartheta) - p_{x_1}(\vartheta)x_2(\vartheta) + z_1(\vartheta)p_{z_2}(\vartheta) - z_2(\vartheta)p_{z_1}(\vartheta) = \text{const.}, \quad \text{(B.1)}$$

which can be put in the form

$$\tilde{\boldsymbol{X}}_2(\vartheta)\mathbf{U}\boldsymbol{X}_1(\vartheta) = \text{const.}, \quad \text{(B.2)}$$

where the tilde $\sim$ indicates the transposed matrix and where

$$\mathbf{U} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad \text{(B.3)}$$

As $X_i(\vartheta) = \mathbf{M}(0|\vartheta) X_i(0)$, we have

$$\tilde{X}_2(0)\tilde{\mathbf{M}}(0|\vartheta)\mathbf{U}\mathbf{M}(0|\vartheta)X_1(0) = \text{const.} = \tilde{X}_2(0)\mathbf{U}X_1(0)\,, \tag{B.4}$$

whatever $X_1(0)$ and $X_2(0)$ may be. From this one obtains the Liapunov–Poincaré relation [16]

$$\tilde{\mathbf{M}}(0|\vartheta)\mathbf{U}\mathbf{M}(0|\vartheta) = \mathbf{U}\,. \tag{B.5}$$

The matrices $\mathbf{M}(0|\vartheta)$, which satisfy condition (B.5), are sometimes called symplectic matrices (ref. 17, ch. 2), as it follows from the symplectic property that the eigenvalues of matrix $\mathbf{M}$ are mutually inverse in pairs. In fact if V_1 and V_k $(k \neq 1)$ are the eigenvectors of the transfer matrix $\mathbf{M}$ corresponding to the eigenvalues λ_1 and λ_k, one has

$$\mathbf{M}V_1 = \lambda_1 V_1 \quad \text{and} \quad \mathbf{M}V_k = \lambda_k V_k\,.$$

As the transfer matrix $\mathbf{M}$ satisfies condition (B.5) we may write the equation

$$(\lambda_1\lambda_k - 1)\tilde{V}_1\mathbf{U}V_k = 0\,. \tag{B.6}$$

Relation (B.6) must hold for any eigenvector V_k. However, the bilinear form $\tilde{V}_1\mathbf{U}V_k$ cannot be equal to zero for all values $k \neq 1$. This would mean, in fact, that the vector $\tilde{V}_1\mathbf{U}$ would be orthogonal to all vectors V_k and coincide with vector $\tilde{V}_1$, as the system of eigenvectors is complete.

It is easy to see from (B.3) that this cannot be the case as there will always exist at least one value $k \neq 1$ such that

$$\lambda_k = \lambda_1^{-1}\,. \tag{B.7}$$

It is of interest to mention that the symplectic condition (B.5) is much stronger than the relation $\det \mathbf{M} = 1$ we had derived and which is, of course, also valid for equations in canonical form. This last relation is nothing else than the expression of Liouville's theorem on the constancy of phase space volume. When the number of degrees of freedom of the problem considered is k, this theorem imposes only one limitation on the $(2k)^2$ elements of matrix $\mathbf{M}$, whereas the symplectic condition imposes $k(2k-1)$ such limitations. For a one-dimensional motion the symplectic condition is simply equivalent to Liouville's theorem.

APPENDIX C. THE BOGOLYUBOV KRYLOV METHOD OF AVERAGES

Many questions in accelerator theory lead to the definition of a vector $\boldsymbol{X}(\tau)$ whose derivative is small by physical necessity.

The treatment of such a quantity depends on the following idea. The derivative $\mathrm{d}\boldsymbol{X}/\mathrm{d}\tau$ being small, it is natural to represent $\boldsymbol{X}(\tau)$ as the sum of a slowly varying part $\boldsymbol{\xi}(\tau)$ and small rapidly oscillating terms (see eq. C.3). If one assumes that these terms give rise only to small oscillations of $\boldsymbol{X}$ around $\boldsymbol{\xi}(\tau)$ and do not influence the motion noticeably on the average, one arrives at an equation valid to a first approximation. The great advantage of the method is the simplicity of the equations obtained after averaging, which can often be integrated by quadratures. Higher order approximations can be derived further in the same way, although this leads usually to rather complicated expressions.

We shall consider here the case where the equation satisfied by $\boldsymbol{X}(\tau)$ has the form

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau} = \mathrm{g}\boldsymbol{Y}(\tau, \boldsymbol{X}), \tag{C.1}$$

g being a small parameter which we introduce here purely formally. It is assumed that the vector function $\boldsymbol{Y}(\tau, \boldsymbol{X})$ possesses an average value

$$\langle \boldsymbol{Y}(\tau, \boldsymbol{X}) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \boldsymbol{Y}(\tau, \boldsymbol{X}) \mathrm{d}\tau \,. \tag{C.2}$$

The asymptotic solution of eq. (C.1) can then be obtained as follows. In a first approximation in terms of g one may write

$$\boldsymbol{X}(\tau) = \boldsymbol{\xi}(\tau) + \mathrm{g}\tilde{\boldsymbol{Y}}(\tau, \boldsymbol{\xi}) \,, \tag{C.3}$$

where $\boldsymbol{\xi}(\tau)$ satisfies the equation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}\tau} = \mathrm{g}\langle \boldsymbol{Y}(\tau, \boldsymbol{\xi}) \rangle \,, \tag{C.4}$$

where the average is to be taken only with respect to the explicit variable τ. The tilde $\sim$ in (C.3) is the integrating operator, defined by

$$\tilde{\boldsymbol{Y}}(\tau, \boldsymbol{\xi}) = \int [\boldsymbol{Y}(\tau, \boldsymbol{\xi}) - \langle \boldsymbol{Y}(\tau, \boldsymbol{\xi}) \rangle] \mathrm{d}\tau \,. \tag{C.5}$$

In a second approximation one finds

$$\boldsymbol{X}(\tau) = \xi(\tau) + g\tilde{\boldsymbol{Y}}(\tau,\xi) + g^2\left(\tilde{\boldsymbol{Y}}(\tau,\xi)\frac{\partial}{\partial\xi}\right)\boldsymbol{Y}(\tau,\xi) - g^2\frac{\partial\tilde{\tilde{\boldsymbol{Y}}}}{\partial\xi}\langle\boldsymbol{Y}(\tau,\xi)\rangle\,, \qquad \text{(C.6)}$$

where $\xi(\tau)$ now satisfies the equation

$$\mathrm{d}\xi/\mathrm{d}\tau = g\langle\boldsymbol{Y}(\tau,\xi+g\tilde{\boldsymbol{Y}}(\tau,\xi))\rangle \approx$$
$$\approx g\langle\boldsymbol{Y}(\tau,\xi)\rangle + g^2\langle(\tilde{\boldsymbol{Y}}(\tau,\xi)\partial/\partial\xi)\boldsymbol{Y}(\tau,\xi)\rangle\,. \qquad \text{(C.7)}$$

In other words, the second order approximation (C.7) can be obtained from the exact equation (C.1) if one introduces in place of $\boldsymbol{X}(\tau)$ its first approximation (C.3), and takes the average with respect to the explicit variable τ while considering ξ as a constant. Higher order approximations can be derived in the same way.

The reader can find further details, as well as the theoretical basis of the method in ref. 23 cited in chapter 2.

APPENDIX D. RADIATION DAMPING DECREMENTS OF BETATRON AND SYNCHROTRON OSCILLATIONS

Let us define a vector with four components $\boldsymbol{X} = \begin{pmatrix} x' \\ x \\ \delta \\ n \end{pmatrix}$, where x and η have the usual meaning, and where $\delta = \Delta E/E$. We shall consider the effect of the emission of radiation as a perturbation, and write eqs. (2.8), (2.17) and (2.9) of ch. 5 in vector form

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\vartheta} = (\mathbf{P}+\boldsymbol{\Gamma})\boldsymbol{X}\,. \qquad \text{(D.1)}$$

Here $\mathbf{P}$ is the matrix of the unperturbed system

$$\mathbf{P} = \begin{pmatrix} 0 & -(1-n)\dfrac{K^2}{K_0^2} & \dfrac{K^2}{K_0^2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{eV}{2\pi E_s}\sin\varphi_s \\ 1 & qK & 0 & 0 \end{pmatrix}. \qquad \text{(D.2)}$$

Matrix $\mathbf{\Gamma}$ describes the perturbations caused by the emission of radiation (the terms which contain Γ) and the variation in energy (the term Γ_1):

$$\mathbf{\Gamma} = \begin{pmatrix} -\Gamma-\Gamma_1, & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\Gamma K(1-2n) & -2\Gamma-\Gamma_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{D.3}$$

We shall look for a solution of vector equation (D.1) in the form

$$\boldsymbol{X} = \sum_{j=x,s} a_j \boldsymbol{f}_j \exp(\mathrm{i}\int \nu_j \mathrm{d}\vartheta) + \text{c.c.} \tag{D.4}$$

where $a_j = a_j(\vartheta)$ are complex amplitudes, which vary slowly as a result of the perturbations, and where $\boldsymbol{f}_j(\vartheta)$ are the Floquet vector functions of the unperturbed system. These functions can, according to (D.2), be written in the form

$$\boldsymbol{f}_x = \begin{pmatrix} f_x' + \mathrm{i}\nu_x f_x \\ f_x \\ 0 \\ f_\sigma \end{pmatrix}; \qquad \boldsymbol{f}_s = \mathrm{i}\left(\frac{\nu_s}{q\alpha}\right)^{\frac{1}{2}} \begin{pmatrix} \psi' \\ \psi \\ 1 \\ -\dfrac{\mathrm{i}q\alpha}{\nu_s} \end{pmatrix} \tag{D.5}$$

Having brought $\boldsymbol{f}_x$ and $\boldsymbol{f}_s$ into the form (D.5), we shall neglect as usual the terms resulting from coupling between betatron and synchrotron oscillations, which give small corrections of the order of $\mathbf{\Gamma}$ to the corresponding frequencies*.

We shall normalize the one-dimensional Floquet function $f_x(\vartheta)$ as usual

$$(f_x' + \mathrm{i}\nu_x f_x) f_x^* - \text{c.c.} = 2\mathrm{i}\,. \tag{D.6}$$

With normalization (D.6) the resulting Wronskian is

$$\mathscr{D} = |\boldsymbol{f}_x, \boldsymbol{f}_x^*, \boldsymbol{f}_s, \boldsymbol{f}_s^*| = -4\,. \tag{D.7}$$

* We note that f_σ, the fourth component of $\boldsymbol{f}_x$, corresponds to the fast phase oscillations resulting from betatron motion (see ch. 4 § 5).

Let us note that the expression for $f_s(\vartheta)$ in (D.5) is valid both for a constant synchrotron oscillation frequency $\nu_s = \Omega/\omega_s$ and for a frequency ω_s which is varying slowly as a result of changes in the energy E_s and the quantity $V \sin \varphi_s$.

Introducing (D.4) in (D.1) we obtain a system of equations for determining the quantities $a_j = a_j(\vartheta)$

$$\sum_{j=x,s} a_j' f_j \exp(\mathrm{i}\int \nu_j \mathrm{d}\vartheta) + \text{c.c.} = \Gamma \left(\sum_{j=x,s} a_j f_j \exp(\mathrm{i}\int \nu_j \mathrm{d}\vartheta) + \text{c.c.} \right) \quad \text{(D.8)}$$

As the elements of the perturbation matrix are relatively small (of the order of Γ) one can make use of the method of averages (ref. 20 of ch. 5), according to which

$$\begin{aligned} a_x' &= u_x a_x \ , \qquad a_s' = u_s a_s \ ; \\ u_x &= \tfrac{1}{4}\langle |\Gamma f_x, f_x^*, f_s, f_s^*| \rangle \ ; \qquad u_s = \tfrac{1}{4}\langle |f_x, f_x^*, \Gamma f_s, f_s^*| \rangle \ . \end{aligned} \quad \text{(D.9)}$$

With the help of formulae (D.2), (D.5) and (D.9) we obtain for the coefficients u_x, u_s the following expressions

$$u_x = -\frac{1}{2\mathrm{i}} \left\langle \Gamma \left\{ f_x^* (f_x' + \mathrm{i}\nu_x f_x) + \right. \right.$$

$$\left. \left. + (1-2n) \frac{K}{K_0} f_x [\psi (f_x^* - \mathrm{i}\nu_x f_x^*) - \psi' f_x^*] \right\} \right\rangle - \tfrac{1}{2}\Gamma_1 \ , \quad \text{(D.10)}$$

$$u_s = -\frac{1}{2} \left\langle \Gamma \left[2 + (1-2n) \frac{K\psi}{K_0} \right] \right\rangle - \tfrac{1}{2}\Gamma_1 \ . \quad \text{(D.11)}$$

From this and from (D.6) we can derive expressions for the decrements $\langle \xi_x \rangle$ and $\langle \xi_s \rangle$:

$$\langle \zeta_x \rangle = -\operatorname{Re} u_x - \tfrac{1}{2}\Gamma_1 = \frac{1}{2} \left\langle \Gamma \left[1 - (1-2n) \frac{K\psi}{K_0} \right] \right\rangle ; \quad \text{(D.12)}$$

$$\langle \zeta_s \rangle = -\operatorname{Re} u_s - \tfrac{1}{2}\Gamma_1 = \frac{1}{2} \left\langle \Gamma \left[2 + (1-2n) \frac{K\psi}{K_0} \right] \right\rangle , \quad \text{(D.13)}$$

where $\operatorname{Re} u_x$ is the real part of u_x. The expressions obtained are exactly the same as (3.10) and (3.14) of chapter 5.

APPENDIX E. EXCITATION OF BETATRON AND SYNCHROTRON OSCILLATIONS BY QUANTUM EMISSION OF RADIATION

To take into account in a uniform manner the effect on betatron and synchrotron oscillations of the discrete character of the emission of radiation, one should supplement eqs. (2.15) and (2.17) of ch. 5, which give $E_s^{-1}(\mathrm{d}\Delta E/\mathrm{d}\vartheta)$ by a term (refs. 11 and 14, ch. 5)

$$w(\vartheta) = \sum_i \frac{\varepsilon_i \delta(\vartheta - \vartheta_i)}{E_s} - \Gamma_s(\vartheta) . \tag{E.1}$$

Here $\delta(\vartheta - \vartheta_i)$ is the delta function and $\varepsilon_i = \hbar\omega_i$ is the energy of the ith photon, emitted at the instant when the particle is at the azimuthal angle ϑ_i. Evidently, $w(\vartheta)$ is a random function of ϑ with a mean value equal to zero. Other changes in (2.15) and (2.17) of ch. 5, due to the discrete character of the radiation, give corrections which are smaller by orders of magnitude, as loss fluctuations act as perturbations when $E \ll E_{\hbar}$.

The system of equations (2.8), (2.17) and (2.19) of ch. 5, supplemented by the term $w(\vartheta)$, becomes a system of inhomogeneous equations, which can be written in vector form, as we have done already in appendix D:

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\vartheta} - (\mathbf{P} + \boldsymbol{\Gamma})\boldsymbol{X} = \boldsymbol{Y}(\vartheta) . \tag{E.2}$$

The vector $\boldsymbol{Y}(\vartheta)$, which represents here the effect of the fluctuation, has the form

$$\boldsymbol{Y}(\vartheta) = w(\vartheta) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} . \tag{E.3}$$

Let us write the solution of (E.3), as we did in appendix D, in the form

$$\boldsymbol{X} = \sum_{j=x,s} a_j \boldsymbol{f}_j(\vartheta) \exp(\mathrm{i}\int \nu_j \mathrm{d}\vartheta) + \mathrm{c.c.} \tag{E.4}$$

We shall find the functions $a_j(\vartheta)$ by variation of the constants and by averaging. The Floquet functions in the zero-th approximation are expressed as before by formulae (D.5). Introducing (E.4) into (E.2) we get for $a_x(\vartheta)$ and $a_s(\vartheta)$ the equations

$$a'_x = u_x a_x - \tfrac{1}{2}\exp(-\mathrm{i}\int v_x \mathrm{d}\vartheta)\, w(\vartheta)\left[(f_x^{*\prime} - \mathrm{i}v_x f_x^*)\psi - f_x^*\psi'\right] ;$$
$$a'_s = u_s a_s - \tfrac{1}{2}\mathrm{i}(q\alpha/v_s)^{\frac{1}{2}} \exp(-\mathrm{i}\int v_s \mathrm{d}\vartheta)\, w(\vartheta) \qquad \text{(E.5)}$$

where u_x and u_s are given by (D.10) and (D.11). The solution of eq. (E.5) has the following form (for zero initial conditions)

$$a_x = -\frac{1}{2}\int_0^{\vartheta} \left[(f_x^{*\prime} - \mathrm{i}v_x f_x^*)\psi - f_x^*\psi'\right] w(\vartheta') \exp\left(-\mathrm{i}\int_0^{\vartheta'} v_x \mathrm{d}\vartheta - \int_{\vartheta'}^{\vartheta} u_x \mathrm{d}\vartheta\right)\mathrm{d}\vartheta' , \qquad \text{(E.6)}$$

$$a_s = -\frac{\mathrm{i}}{2}\int_0^{\vartheta} \left(\frac{q\alpha}{v_s}\right)^{\frac{1}{2}} w(\vartheta') \exp\left(-\mathrm{i}\int_0^{\vartheta'} v_s \mathrm{d}\vartheta - \int_{\vartheta'}^{\vartheta} u_s \mathrm{d}\vartheta\right)\mathrm{d}\vartheta' . \qquad \text{(E.7)}$$

As the mean value $\bar{w} = 0$, the mean probable values $\bar{a}_x$ and $\bar{a}_s$ are also equal to zero. We are, however, interested in the mean square amplitudes $\overline{A_x^2}$ and $\overline{A_\eta^2}$ which are related to $a_{x,s}$ by

$$\overline{A_x^2} = 4|f_x|^2_{\max}\,\overline{|a_x|^2} ; \qquad \overline{A_\eta^2} = \frac{4q\alpha}{v_s}\,\overline{|a_s|^2} . \qquad \text{(E.8)}$$

Let us start by determining the quantity $\overline{|a_x|^2}$. As follows from (E.6)

$$\overline{|a_x|^2} = \frac{1}{4}\int_0^{\vartheta}\int_0^{\vartheta} \left[(f_x^{*\prime} - \mathrm{i}v_x f_x^*)\psi - f_x^*\psi'\right]_{\vartheta'} \left[(f'_x + \mathrm{i}v_x f_x)\psi - f_x\psi'\right]_{\vartheta''} \times$$
$$\times\, \overline{w(\vartheta')w(\vartheta'')} \exp\left(-\int_{\vartheta''}^{\vartheta'} \mathrm{i}v_x \mathrm{d}\vartheta - \int_{\vartheta'}^{\vartheta} u_x \mathrm{d}\vartheta - \int_{\vartheta''}^{\vartheta} u_x^* \mathrm{d}\vartheta\right)\mathrm{d}\vartheta'\mathrm{d}\vartheta'' , \qquad \text{(E.9)}$$

where the indices of the square brackets mean that they refer to azimuths ϑ' and ϑ'', respectively. We have indicated statistical averages everywhere by a bar.

To calculate $\overline{w(\vartheta')w(\vartheta'')}$ let us note first that the emission probability of a photon of the jth kind in the interval $(\vartheta_i, \vartheta_i + \mathrm{d}\vartheta_i)$ is equal to $\bar{n}_j(\vartheta_i)\mathrm{d}\vartheta_i$. It follows, because of the statistical independence of single emission processes, that

$$\overline{\sum_i \frac{\varepsilon_i}{E}\,\delta(\vartheta-\vartheta_i)} = \overline{\sum_j \frac{\varepsilon_j}{E}\,\bar{n}_j(\vartheta)} = \Gamma(\vartheta)\ ;$$

$$\overline{\sum_{i\neq k} \frac{\varepsilon_i\varepsilon_k}{E^2}\,\delta(\vartheta'-\vartheta_i)\delta(\vartheta''-\vartheta_k)} = \Gamma(\vartheta')\Gamma(\vartheta'')\ ;$$

$$\overline{\sum_i \frac{\varepsilon_i^2}{E^2}\,\delta(\vartheta'-\vartheta_i)\delta(\vartheta''-\vartheta_i)} = \sum_j \frac{\varepsilon_j^2}{E^2}\,\bar{n}_j(\vartheta')\delta(\vartheta'-\vartheta'')\,. \tag{E.10}$$

According to the definition of function $w(\vartheta)$ by (E.1), and to (5.7) of ch. 5, we obtain

$$\overline{w(\vartheta')\,w(\vartheta'')} = \sum_j \frac{\varepsilon_j^2}{E^2}\,\bar{n}_j(\vartheta')\delta(\vartheta'-\vartheta'') = \frac{55}{24\sqrt{3}}\,\frac{\Lambda r_e K^3(\vartheta')}{K_0}\,\gamma^5\,\delta(\vartheta'-\vartheta'')\ ; \tag{E.11}$$

where Λ is the Compton wavelength and r_e the classical radius of the electron.

Let us now introduce (E.11) into (E.9) and integrate* with respect to ϑ''. We obtain

$$\overline{|a_x|^2} = \frac{55\Lambda r_e}{96\sqrt{3}K_0}\int_0^{\vartheta} |(f_x'+\mathrm{i}v_x f_x)\psi - f_x\psi'|^2\,K^3\gamma^5 \exp\left(\int_{\vartheta'}^{\vartheta} 2\,\mathrm{Re}\,u_x\,\mathrm{d}\vartheta''\right)\mathrm{d}\vartheta'\,. \tag{E.12}$$

If we consider that, according to (D.12),

$$\mathrm{Re}\,u_x = \tfrac{1}{2}\Gamma_1 + \langle\zeta_x\rangle\,, \tag{E.13}$$

and remove from under the integration sign the mean value of the fast oscillating terms, we obtain

$$\overline{A_x^2} = 4|f|^2_{\max}\,\overline{|a_x|^2} = \frac{55}{24\sqrt{3}}\,\frac{r_e\Lambda F}{\gamma}\int_0^{\vartheta}\gamma^6 \exp\left[-2\int_{\vartheta'}^{\vartheta}\langle\zeta_x\rangle\,\mathrm{d}\vartheta''\right]\mathrm{d}\vartheta'\,, \tag{E.14}$$

where factor F is given by (5.9) of ch. 5.

* According to (E.11), the correlation function $\overline{w(\vartheta')w(\vartheta'')}$ of the $w(\vartheta)$ process is proportional to $\delta(\vartheta''-\vartheta')$. This means that $w(\vartheta)$ is white noise with a spectral density

$$\Phi = \frac{55}{12\sqrt{3}}\,\frac{\Delta r_e K^3}{K_0}\,\gamma^5\,.$$

The quantity $\overline{A_\eta^2}$ can be calculated in a similar manner. In fact, we obtain from (E.7) and (E.11)

$$\overline{|a_s|^2} = \frac{55\Lambda r_e q\alpha\langle K^3\rangle}{96\sqrt{3}\,K_0}\int_0^{\vartheta}\frac{\gamma^5}{\nu_s}\exp\left[-2\int_{\vartheta'}^{\vartheta}\mathrm{Re}\,u_s\,\mathrm{d}\vartheta''\right]\mathrm{d}\vartheta' \qquad \text{(E.15)}$$

or, if we introduce u_s from (D.13) and replace ν_s by its expression

$$\overline{A_\eta^2} = \frac{55\pi}{12\sqrt{3}}\,e\Lambda q\alpha\,\frac{\langle K^3\rangle}{K_0}\,(\gamma V\sin\varphi_s)^{-0.5}\int_0^{\vartheta}\gamma^{6.5}(V\sin\varphi_s)^{-0.5}\times$$

$$\times\exp\left[-2\int_{\vartheta'}^{\vartheta}\langle\zeta_s\rangle\,\mathrm{d}\vartheta''\right]\mathrm{d}\vartheta'\,. \qquad \text{(E.16)}$$

It is easily seen that (E.14) and (E.16) are identical to (5.13) and (5.23) of ch. 5, which had been derived from relatively simple considerations.

APPENDIX F. PARTICLE MOTION IN AN AZIMUTHALLY SYMMETRICAL MAGNETIC FIELD

Charged particle motion in an azimuthally symmetrical magnetic field is one of the rare cases where it is possible to determine the trajectories for any initial conditions, and not only for small deviations from a given orbit.

Let us start from the Lagrange function

$$L = -m_0c^2\sqrt{1-\beta^2} + \frac{e}{c}(vA)\,, \qquad \text{(F.1)}$$

where the vector potential of the azimuthally symmetrical magnetic field has only an azimuthal component

$$A = A_\theta = \frac{1}{r}\int_0^r B_z(r,z,t)\,r\,\mathrm{d}r\,. \qquad \text{(F.2)}$$

From (F.1) and (F.2) we derive the equations of motion as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = mr\dot{\theta}^2 + \frac{e}{c} r\dot{\theta}B_z\,, \tag{F.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta} + \frac{e}{c}\int_0^r B_z r\,\mathrm{d}r\right) = 0\,, \tag{F.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z}) = -\frac{e}{c} B_r r\dot{\theta}\,. \tag{F.5}$$

The physical meaning of the terms which appear in (F.3) and (F.5) is evident. The sense of the angular momentum equation (F.4) is easily understood with the help of the relations

$$B_r = -\frac{1}{r}\int_0^r \frac{\partial B_z}{\partial r} r\,\mathrm{d}r\,, \quad \mathscr{E}_\theta = \mathscr{E} = \frac{1}{cr}\int_0^r \frac{\partial B_z}{\partial t} r\,\mathrm{d}r\,, \tag{F.6}$$

(where $\mathscr{E}_\theta$ is the rotational component of the electric field), if we rewrite it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = -e\mathscr{E}_\theta r - \frac{e}{c} B_z r\dot{r} + \frac{e}{c} B_r r\dot{z}\,. \tag{F.7}$$

Equations (F.4) and (F.7) thus appear to be the law of conservation of the 'generalized' angular momentum P

$$P = mr^2\dot{\theta} + \frac{e}{c}\int_0^r B_z r\,\mathrm{d}r = \text{const.} \tag{F.8}$$

It is easy to see that if the initial conditions t_i, r_i, p_i (p_i is the particle momentum) are given, P determines the distance of the initial line of flight from the centre of the field. With (F.8) one can modify (F.3) and (F.5) as follows

$$\text{(a)}\quad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = -\frac{p_\theta}{m}\frac{\partial p_\theta}{\partial r}\,; \qquad \text{(b)}\quad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z}) = -\frac{p_\theta}{m}\frac{\partial p_\theta}{\partial z}\,, \tag{F.9}$$

where $p_\theta = mr\dot{\theta}$ is the angular momentum of the particle.

When the field does not vary with time (that is when the mass m is also a constant, which can be different from the rest mass, $m \neq m_0$), these equations can be written in a way which corresponds to the motion in r- and z-directions with a potential energy

$$U = \frac{p_\theta^2}{2m} = \frac{1}{2m}\left[\frac{P}{r} - \frac{e}{c}\frac{1}{r}\int_0^r B_z(z,r) r\,\mathrm{d}r\right], \tag{F.10}$$

$$\text{(a)}\quad m\ddot{r} = -\frac{U}{\partial r}\,; \qquad \text{(b)}\quad m\ddot{z} = -\frac{\partial U}{\partial z}\,. \tag{F.11}$$

The characteristic of function $U = U(r, z, P)$, which may be called the 'quasi-potential', is its dependence on the parameter P defined by the combination of the initial conditions (F.8). When P is given, one can obviously use the law of conservation of energy. The quantity $W = (p_r^2 + p_z^2)/2m$ corresponding to the motion in the meridian plane r, z plays the part of kinetic energy and the function $U = p_\theta^2/2m$ corresponding to the motion in the azimuthal direction, the part of the potential energy. Thus, one has

$$W + U = \text{const.} \tag{F.12}$$

In addition to azimuthal rotational symmetry, let us assume the existence of a symmetry plane and examine the motion in this plane $z = 0$. The relations $\mathrm{d}\theta/\mathrm{d}r = p_\theta/rp_r$ and (F.10) then yield the ray equation $\theta = \theta(r)$, which is determined by three constants r_i, p, P.

$$\theta = \int_{r_i}^{r} \frac{\left(P - \frac{e}{c}\int_0^\xi B\xi\,\mathrm{d}\xi\right)\mathrm{d}\xi}{\pm\xi^2\left[p^2 - \xi^{-2}\left(P - \frac{e}{c}\int_0^\xi B\xi\,\mathrm{d}\xi\right)^2\right]^{\frac{1}{2}}} + \theta_i\,. \tag{F.13}$$

Characteristic points C of the trajectory are those for which the radius vector, after having increased, begins to decrease, or vice versa. For these we have

$$(\mathrm{d}r/\mathrm{d}\theta)_{r=r_C} = 0\,, \qquad p_\theta(r_C) = \pm p\,. \tag{F.14}$$

Let us now indicate a few special cases, in which the integral in

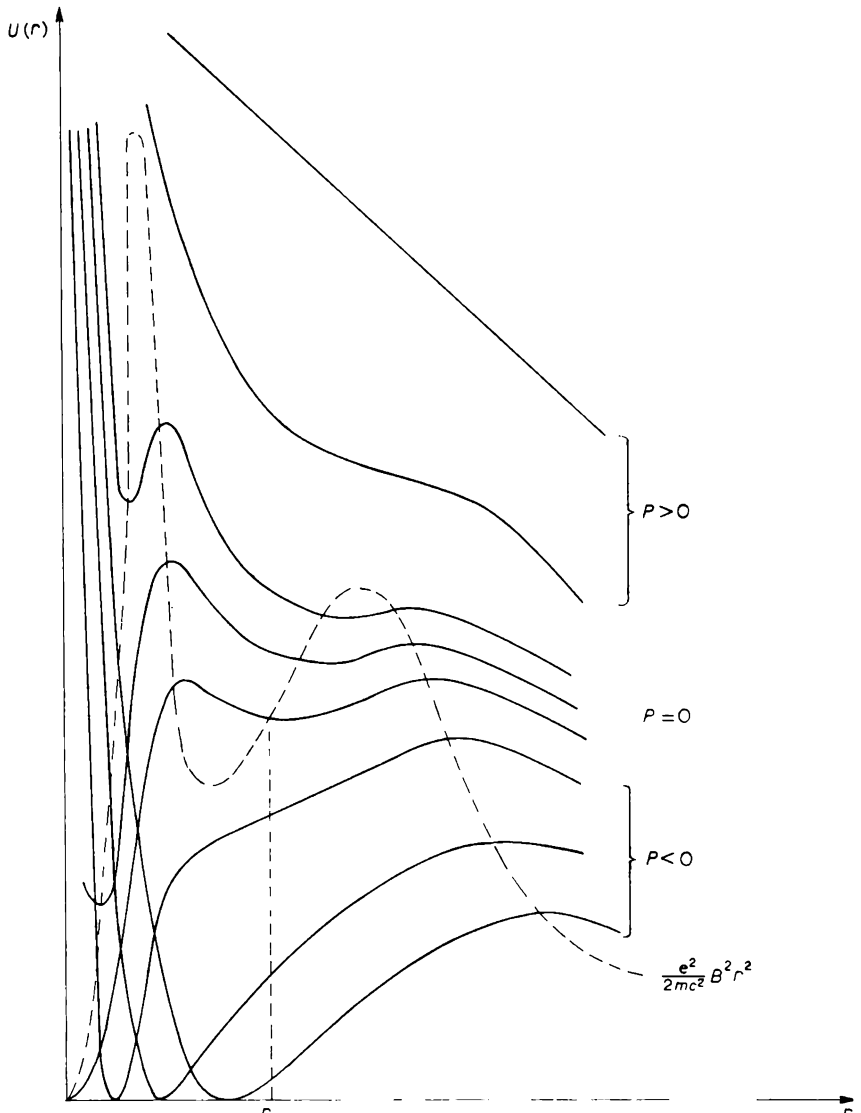

Fig. 85. Dependence of the effective potential on radius for various initial conditions (schematic).

(F.13) diverges [this corresponds to $(\mathrm{d}U/\mathrm{d}r)_{r=r_C}=0$]. In this case the motion can take place either along a circle, or along a 'coiled' trajectory which tends asymptotically towards a circle, winding round it. The coiled trajectory is always unstable, that is the boundary point r_0 corresponds to the maximum of the function $U(r, P)$. The motion along a circle, which is of interest for accelerators, may be stable or unstable. To such a motion with $r=r_0=$ const. corresponds an extremum of the potential function

$$(\mathrm{d}U/\mathrm{d}r)_{r=r_0}=0\,, \tag{F.15}$$

which yields, with (F.8), the evident relation

$$p=p_0=(e/c)B(r_0)r_0\,. \tag{F.16}$$

The stable motion along the circle corresponds to the minimum of $U(r, P)$, so that, besides (F.15), it must satisfy

$$(\mathrm{d}^2U/\mathrm{d}r^2)_{r=r_0}>0\,,\ \text{that is}\ n<1\,. \tag{F.17}$$

One sees at once that the geometric locus of the extrema of the potential curves in the $r, U(r, P)$ plane will be the curve $(e/2mc^2)B_z^2(r)r^2$, the minima lying on the rising parts of this curve (for $n<1$) and the maxima on the falling parts (for $n>1$). An example of a family of potential curves for the rotationally symmetrical field of a betatron is given in fig. 85.

We notice that the extrem of the potential curves corresponding to $P=0$ all lie at a point r_b where the '2 to 1' condition is fulfilled, that is

$$B_z(r_b)=\tfrac{1}{2}\bar{B}_z(r_b)=r_b^{-2}\int_0^{r_b}B_z r\,\mathrm{d}r\,. \tag{F.18}$$

With the concept of the 'quasi-potential', one can thus study the various types of trajectories possible in a rotationally symmetrical field: they may be finite or infinite, with or without crossover points (trochoids). Difficulties of principle are apparently not to be expected when investigating trajectories in the non-linear approximation in terms of excursions away from some given curve.

APPENDIX G. CALCULATION OF ORBITS AND BETATRON OSCILLATIONS IN THE RING-PHASOTRON

To obtain an approximate expression for the plane closed orbit in the ring-phasotron, one can use the ray equation (3.22a) in ch. 7 for $z \equiv 0$

$$r'' = \frac{1}{r}[r^2 + 2r'^2] + \frac{1}{r}[r^2 + r'^2]^{\frac{3}{2}} \frac{eB(r,\theta)}{pc}, \tag{G.1}$$

and expand its right-hand side in powers of ξ, ξ' with $r = R(1+\xi)$. Taking into account the fact that the magnetic field in the median plane depends critically on the co-ordinates r and θ, and that the orbit oscillations around the mean value have a small period for $N \gg 1$, we shall write the various terms in order of decreasing magnitude, as reflected formally by the powers of the parameter $\mathfrak{g}$

$$\xi \sim \lambda_0 \mathfrak{g}^2 \; ; \qquad \xi' \sim \lambda_0 \mathfrak{g} \; ; \qquad \xi'' \sim \lambda_0 \; ;$$

$$\frac{R}{B_0} \frac{\partial B}{\partial r}\bigg|_{r=R} \sim \mathfrak{g}^{-1} \; ; \qquad \frac{R^2}{B_0} \frac{\partial^2 B}{\partial r^2}\bigg|_{r=R} \sim \mathfrak{g}^{-2} \quad \text{etc.} \tag{G.2}$$

$$B_0 = \max B(R,\theta) \, .$$

Depending on the form of the magnetic field, the parameter $\lambda_0 = eB_0R/pc$ can have different orders of size. In the one-way machine, where the magnetic field has a large constant component, the circumference factor is near to one, in other words $\lambda_0 \sim \mathfrak{g}^0$. In the two-way machine, where λ_0 has a large value, $\lambda_0 \sim \mathfrak{g}^{-\frac{1}{2}}$. The correctness of these assumptions and the physical significance of the parameter $\mathfrak{g}$ we have used for expanding will be clear from the results obtained below.

With relations (G.2) we obtain the following expansion of eq. (G.1)

$$\xi'' = 1 + \lambda_0 \frac{B}{B_0} + \xi\left[1 + 2\lambda_0 \frac{B}{B_0} + \lambda_0 \frac{R}{B_0} \frac{\partial B}{\partial r}\right] +$$

$$+ \xi'^2\left[2 + \tfrac{3}{2}\lambda_0 \frac{B}{B_0}\right] + \tfrac{1}{2}\xi^2 \lambda_0 \frac{R^2}{B_0} \frac{\partial^2 B}{\partial r^2} + \ldots, \tag{G.3}$$

where the terms neglected have the order $\mathfrak{g}^3$ for the one-way and $\mathfrak{g}^{\frac{3}{2}}$ for the two-way machine. Let us introduce here the Fourier series for

the magnetic field and its derivatives

$$\frac{B}{B_0} = \sum b_k \mathrm{e}^{\mathrm{i}kN\theta},$$

$$\frac{R}{B_0}\frac{\partial B}{\partial r} = -\sum b_k[n_0+\mathrm{i}kNh]\mathrm{e}^{\mathrm{i}kN\theta}, \tag{G.4}$$

$$\frac{R^2}{B_0}\frac{\partial^2 B}{\partial r^2} = \sum b_k[n_0+\mathrm{i}kNh][n_0+1+\mathrm{i}kNh]\mathrm{e}^{\mathrm{i}kN\theta}$$

and let us look for a solution $\xi(\theta)$ also in the form of a Fourier series as indicated in eq. (3.31) of ch. 7. We then obtain to a first approximation for the coefficients ξ_k

$$\xi_k = -\frac{\lambda_0 b_k}{k^2 N^2}, \qquad k \neq 0. \tag{G.5}$$

From the condition $\xi = 0$, that is assuming R to be the mean orbit radius, we obtain from (G.5) an equation to determine λ_0:

$$0 = 1+\lambda_0 b_0 + \frac{\lambda_0^2}{N^2}\sum_{k\neq 0}\frac{|b_k|^2}{k^2}(n_0-\mathrm{i}kNh)+\ldots \tag{G.6}$$

The terms neglected have the order g^2 for the one-way and g for the two-way machine. The method of deriving equations to a higher order of approximation for λ_0 is evident.

One can express the sum in eq. (G.6) by the quantity S^2 defined in formula (3.30) of ch. 7, and write the equation in the form

$$1+\lambda_0 b_0 + \frac{\lambda_0^2 n_0 S^2}{N^2} = 0. \tag{G.7}$$

Its solution is (see eq. 3.33 of ch. 7)

$$\lambda_0 = \frac{N^2}{2n_0 S^2}\left[-b_0 \pm \left(b_0^2 - \frac{4n_0 S^2}{N^2}\right)^{\frac{1}{2}}\right]. \tag{G.8}$$

This gives in the asymmetrical case (one-way machine)

$$\lambda_0 \approx -\frac{1}{b_0} - \frac{S^2 n_0}{N^2 b_0^2} + \ldots, \tag{G.9}$$

and in the nearly symmetrical case $\left(b_0^2 \ll \frac{4|n_0|}{N^2} S^2\right)$:

$$\lambda_0 = \pm \frac{N}{S\sqrt{-n_0}} \left[1 \pm \frac{b_0 N}{2S\sqrt{-n_0}} - \frac{b_0 N^2}{8 n_0 S^2} + \ldots\right]. \tag{G.10}$$

If one compares the expressions obtained with the conditions (G.2) it is easy to see that the role of the order of magnitude parameter $\varrho \ll 1$ is played in fact by $1/N$ and usually by $1/n_0$, so that a rapid convergence of all approximate formulae of this section is ensured only when $1 \; N \ll 1$.

Let us now determine the frequencies of radial and vertical betatron oscillations. For this we shall use the averaging method, as shown in § 3 of ch. 7. Let us linearize the initial equations of radial and vertical motion (3.22a, b), in terms of small excursions z and x from the closed orbit, whereby

$$x = r - R(1 - \xi(\theta)) . \tag{G.11}$$

We obtain

$$\begin{aligned} x'' - 2\Gamma_x x' + g_x x = 0 , \\ z'' - 2\Gamma_z z' + g_z z = 0 , \end{aligned} \tag{G.12}$$

where

$$\begin{aligned} \Gamma_x &\approx \xi' \left[2 + \tfrac{3}{2}\lambda_0 \frac{B}{B_0}\right], \\ \Gamma_z &\approx \tfrac{1}{2}\xi' \left[2 + \lambda_0 \frac{B}{B_0}\right], \\ g_x &\approx -\left(1 + 2\lambda_0 \frac{B}{B_0} + \dot{\lambda}_0 \frac{R}{B_0} \frac{\partial B}{\partial r}\right) - \xi \dot{\lambda}_0 \frac{R^2}{B_0} \frac{\partial^2 B}{\partial r^2} + \ldots , \\ g_z &\approx \lambda_0 \frac{R}{B_0} \frac{\partial B}{\partial r} + \xi \lambda_0 \frac{R^2}{B_0} \frac{\partial^2 B}{\partial r^2} - \xi' \lambda_0 \frac{1}{B_0} \frac{\partial B}{\partial \theta} + \ldots \end{aligned} \tag{G.13}$$

The factors $\Gamma_{x,z}$ are equal to zero on the average and do not cause damping of the oscillations. In principle, they bring small corrections to the expressions for the frequencies, but may be neglected in our approximation.

With formulae (G.13) we obtain the following approximate expressions for the Fourier coefficients g_0 and g_k

$$
\begin{aligned}
g_{x0} &\approx \lambda_0 n_0 b_0 - (1+2\lambda_0 b_0) + \frac{\lambda_0^2 n_0^2}{N^2} S^2 - \lambda_0^2 h^2 \Phi^2 + \dots , \\
g_{z0} &\approx -\lambda_0 n_0 b_0 - \frac{\lambda_0^2 n_0^2}{N^2} S^2 + \lambda_0^2 \Phi^2 (h^2+1) + \dots , \\
g_{xk} &\approx \lambda_0 (n_0 + \mathrm{i}Nhk) b_k + \dots ; \\
g_{zk} &\approx -\lambda_0 (n_0 + \mathrm{i}Nhk) + \dots .
\end{aligned}
\tag{G.14}
$$

Introducing (G.14) in the formula for the frequencies (see eq. 7.16 of ch. 2) we obtain

$$
\nu^2 = \left\langle \left[(g - \langle g \rangle) \mathrm{d}\theta \right]^2 + g \right\rangle \equiv 2 \sum_1^\infty \frac{|g_k|^2}{k^2 N^2} + g_0 \tag{G.15}
$$

and with eq. (G.7), we find

$$
\begin{aligned}
\nu_x^2 &\approx 1 - n_0 + \frac{\lambda_0^2 n_0^2 S^2}{N^2} + \dots , \\
\nu_z^2 &\approx n_0 + \frac{\lambda_0^2 n_0^2 S^2}{N^2} + \lambda_0^2 \Phi^2 (2h^2+1) + \dots .
\end{aligned}
\tag{G.16}
$$

In particular we shall have for the asymmetrical machine ($\lambda_0 \approx -1/b_0$):

$$
\begin{aligned}
\nu_x^2 &\approx 1 - n_0 + \frac{n_0^2 S^2}{N^2 b_0^2} + \dots , \\
\nu_z^2 &\approx n_0 + \frac{n_0^2 S^2}{N^2 b_0^2} + \frac{\Phi^2}{b_0^2} (2h^2+1) + \dots ,
\end{aligned}
\tag{G.17}
$$

and

$$
\begin{aligned}
\nu_x^2 &\approx -2n_0 + \dots , \\
\nu_z^2 &\approx -\frac{N^2 \Phi^2}{n_0 S^2} (2h^2+1) + \dots
\end{aligned}
\tag{G.18}
$$

for the symmetrical one $\left(\lambda_0 \approx \pm \frac{N}{S\sqrt{-n_0}}, \quad |n_0| \gg 1 \right)$.

BIBLIOGRAPHY

CHAPTER I

1. Lawrence E. O., Livingston M. S., Phys. Rev. **37,** 1707 (1931).
2. Rukavishnikov V. N., Phys. Rev. **52,** 1077 (1937).
3. Wideröe R., Arch. f. Elektr. **21,** 387 (1928).
4. Kerst D. W., Phys. Rev. **58,** 841 (1940).
5. Veksler V. I., DAN SSSR **43,** 346; **44,** 393 (1944); Journ. Phys. **9,** 153 (1945); Phys. Rev. **69,** 244 (1946).
6. MacMillan E. M., Phys. Rev. **68,** 143 (1954).
7. Christofilos N., A System to focus ions and electrons and its application to magnetic resonance accelerators (not published) (1950).
8. Courant E. D., Livingston M. S., Snyder H. S., Phys. Rev. **88,** 1190 (1952). (Russian translation in PSF, No. 11, 169 (1954)).
9. Regenstreif E., CERN Report 59-29, Geneva (1959).
10. Green G. K., Proceedings of the International Conference on Accelerators, CERN, Geneva, 347 (1959).
11. Vladimirski V. V., Komar E. G., Mints A. L., et al., AE 4, 31 (1956); CERN Symposium **1,** 118, 122 (1956).
12. Thomas C. H., Phys. Rev. **54,** 580 (1938).
13. Heyn F. A., and Tat K. K., Rev. Sci. Instr. **29,** 662 (1958).
14. Vasilevskaya D. P., Glazov A. A., Danilov V. N., Denisov Yu. N., Dzhelepov V. P., Dmitrievski V. P., Zamolodchikov B. I., Zaplatin N. L., Kolga V. V., Kropin A. P., Lyu Ne-guan, Rybalko V. S., Savenkov A. L., Sarkisian L. A., AE **6,** 657 (1959); **8,** 189 (1960).
15. Kolomensky A. A., Petukhov V. A., Rabinovich M. S., Manual "Some questions on the theory of cyclic accelerators", page 7, Edition A.N. SSSR (1955); PTE, No. 2, 26 (1956).
16. Symon K. R., Kerst D. W., Jones L. W., Terwilliger K. M., Laslett L. J., Phys. Rev. **103,** 1837 (1956). (Russian translation in UFN **61,** 613 (1957)).
17. Kerst D. W., CERN Symposium **1,** 36 (1956).
18. O'Neill G. K., Proceedings of the International Conference on Accelerators, CERN, Geneva, page 125 (1959).
19. Kolomensky A. A., ZhETF **33,** 298 (1957); Ohkawa T., Rev. Sci. Instr. **29,** 108 (1958).
20. Budker G. I., CERN Symposium **1,** 68 (1956); AE **1,** No. 5, 9 (1956).
21. Veksler V. I., CERN Symposium **1,** 80 (1956); AE **2,** 427 (1957).
22. Fajnberg Ya. B., CERN Symposium **1,** 84 (1956); AE **6,** 431 (1959).
23. Veksler V. I., Kolomensky A. A., Petuchov V. A., Rabinovich M. S., Appendix No. 4 to AE **5,** (1957).

24. Landau L., and Lifshits E., Field Theory, § 10, Fizmatgiz, 1960.
25. Petukhov V. A., ZhETF **32,** 379 (1957).
O'Neill G. K., Barber W. C., Burton Richter, Panofski W. K. H., "A Proposed Experiment on the Limits of Quantum Electrodynamics". Stanford University Report (1958).
26. Kerst D. W., and Serber R., Phys. Rev. **60,** 53 (1941).
27. Terletsky Ya P., ZhETF **11,** 96 (1941); Journ. Phys. **9,** 159 (1945).
28. Tamm I. E., Bases of the Theory of Electricity, § 19, Gostekhizdat, 1946.
29. Chandrasekhar S., Stochastic Problems in Physics and Astronomy, Chapter III. § 4, IL, 1947.

CHAPTER II

1. Courant E. D., Snyder H. S., Ann. of Phys. **3,** 1 (1958). Russian translation in PSF, No. 4, 91 (1958).
2. Rashevsky P. K., A Course on Differential Geometry, Gostekhizdat, 1956.
3. Lyapunov A. M., The general problem of the stability of motion, Gostekhizdat, 1950.
4. Sansone G., Ordinary Differential Equations, IL, 1953 and Equazioni differenziali nel campo reale, Bologna 1948.
5. Malkin I. G., The Theory of the Stability of Motion, Gostekhizdat, 1952.
6. Grinberg G. A., Selected Questions of the Mathematical Theory of Electric and Magnetic Phenomena, Edition AN SSSR, 1948.
7. Kolomensky A. A., ZhTF **26,** 1969, 1978 (1956). Research on the Theory of Motion of Particles in Modern Cyclic Accelerators. Thesis, FIAN (1957); Proceedings of the FIAN **XIII,** 3 (1960).
8. Stratton J. A., The Theory of Electromagnetism, MacGraw Hill 1941 and Gostekhizdat, 1948.
9. Smirnov V. I., Course of higher mathematics, Gostekhizdat, 1957.
10. Stoker J. J., Nonlinear Oscillations in Mechanical and Electrical Systems, New-York, Interscience, 1950 and IL, 1953.
11. MacLachlan N., Theory and Application of the Mathieu Function, Clarendon Press, Oxford, 1951 and IL, 1953.
12. Kapitza P. L., ZhETF **21,** 5 (1951); UFN XLIV, 7 (1951).
13. Baldin A. M., Mikhajlov V. V., Rabinovich M. S., ZhETF **31,** 993 (1956).
14. Goldstein G., Classical Mechanics, Addison-Wesley Cambridge, 1950 and Gostekhizdat, 1957.
15. Seiden J., Ann. de Phys. **10,** 259 (1955) [Russian translation in PSF, No. 4, 39 (1956)].
16. Poincaré, Mécanique Céleste **1,** 193 (1892).
17. Weyl G., Classical Groups, Invariants and Representations, Princeton 1946 and IL, 1947.
18. Hereward H. et al., CERN Symposium **1,** 179 (1956).
19. Kolomensky A. A., Kuznetsov A. B., Rubin N. B., ZhTF **29,** 981 (1959).
20. Landau L. D., and Lifshits E. M., Mechanics, Oxford, Pergamon 1960 and Fizmatgiz, 1958.

21. Born M., Lectures on Atomic Mechanics, t. 1, ONTI, 1934 and Atomic Physics, Hafner New-York, 1957.
22. Krylov N. M., and Bogolyubov N. N., New Methods of Non-linear Mechanics, GTTI, 1934.
23. Bogolyubov N. N., and Mitropolskij Yu. A., Asymptotic Methods in the Theory of Non-linear Oscillations, Fizmatgiz, 1958.
24. Whittaker E., and Watson G., Course on Modern Analysis, Cambridge University Press 1946 and GTTI, 1933.
25. Proskuryakov A. P., Applied Mathematics and Mechanics **10,** 545 (1946).
26. Hill, Acta Mathematica **8,** 1 (1886).
27. Adler F., Barconcini D., Nuovo Cimento **IV,** 959 (1956).
28. Symon K. et al., Phys. Rev. **103,** 1837 (1956). [Russian translation UFN, **XI,** 613 (1957)].

CHAPTER III

1. Dennison D, and Berlin T., Phys. Rev. **69,** 542; **70,** 764 (1946).
2. Courant E. D., Journ. Appl. Phys. **20,** 611 (1949).
3. Kolomensky A. A., and Rabinovich M. S., On a new accelerator for high energies, Review FIAN, RF-62 (1952).
4. Vladimirsky V. V., Research on the stability of the trajectory in the strong-focusing synchrotron, Review AN USSR (1953).
5. Rabinovich M. S., Theoretical investigations on the accelerator with variable magnetic field index, FIAN Report (1953); Basis of the theory of the Synchrophasotron with straight sections, Proceedings FIAN X, 23 (1958).
6. Kolomensky A. A., ZhTF **26,** 1969, 1978 (1956); Proceedings FIAN XIII, 3 (1960).
7. Adams J. B., Hine M. G., Lawson J. D., Nature **171,** 926 (1953).
8. Sands M, and Touschek B., Nuovo Cimento **10,** 604 (1953).
9. Lüders G., Nuovo Cimento Suppl. **2,** 1075 (1955).
10. Bogolyubov N. N., and Mitropolskij Yu. A., Asymptotic methods in the theory of non-linear oscillations, Fizmatgiz, 1958.
11. Courant E. D., Snyder H. S., Ann. of Phys. **3,** 1 (1958). [Russian translation in PSF, No. 4, 91 (1958)].
12. Courant R. and Hilbert D., Methods of mathematical physics, New-York, Interscience, 1953 and Gostekhizdat, 1951.
13. Orlov Yu. F., ZhETF **32,** 316 (1957); Non-linear theory of betatron oscillations in synchrotrons with strong focusing, Thesis Yerevan (1958).
14. Orlov Yu. F., and Tarasov E. K., Proceedings of the International Conference on Accelerators, CERN, Geneva, 263 (1959).
15. Moser J., Nachr. Acad. Gött. IIa, No. 6, 87 (1955).
16. Sturrock P. A., Ann. of Phys. **3,** 113 (1958).
Sturrock P. A., Statistical and dynamical electron optics, IL, 1958.
17. Schoch A., CERN Report, 57-21 (1958).
18. Hagedorn R., CERN Report, 57-1 (1957).
Hagedorn R., Schoch A., CERN Report 57-14 (1957).

19. Kolomensky A. A., CERN Symposium **1,** 265 (1956); ZhTF **26,** 1969, 1978 (1956).
20. Andronov A. A., Vitt A. A., Khajkin S. E., Theory of oscillations, Fizmatgiz, 1959.
21. Bulgakov B. V., Oscillations, Gostekhizdat, 1954.
22. Tihiro Hayashi, Forced oscillations in non-linear systems, IL, 1957.
23. Chirikov B. V., Non-linear oscillations in nearly linear systems, Thesis, Novosibirsk (1959).
24. Kotov V. I., and Rubin N. B., ZhTF **28,** 365 (1958).

CHAPTER IV

1. Sansone G., Ordinary Differential Equations, IL, 1953 and Equazioni differenziali nel campo reale, Bologna, 1948.
2. Courant E. D., and Snyder H. S., Ann. of Phys. **3,** 1 (1958). [Russian translation in PSF, No. **4,** 91 (1958)].
3. Rabinovich M. S., Journ. Phys. USSR **10,** 523, 530 (1946). Theory of modern resonance accelerators, FIAN (1948).
4. Bohm D., and Foldy L., Phys. Rev. **70,** 249 (1946); **72,** 649 (1947). [Russian translation in the manual "Resonance cyclic accelerators of elementary particles", IL, (1950)].
5. Frank N. H., Phys. Rev. **70,** 177 (1946).
6. Goldin L. L., and Koshkarev D. G., ZhETF **31,** 803 (1956).
7. Lebedev A. N., FIAN Report (1960); PTE No. 6, 21 (1961).
8. Vorob'ev A. A., Didenko A. N., Kovalenko E. C., and Morozov V. N., Proceedings of the International Conference on Accelerators, CERN, Geneva, 680 (1959).
9. Klima R., Czech Journ. of Phys. **10,** 136 (1960).
10. Landau L. D., and Lifshits E. M., Mechanics, Fizmatgiz, 1958.
11. Kaiser T. R., Proc. Phys. Soc. **63A,** 52 (1950). [Russian translation in PSF, No. 2, 61 (1952)].
12. Danilkin I. S., Pisarev V. E., AE 4, No. 6 (1958); Danilkin I. S., Particular Regimes in the operation of resonance cyclic accelerators, Thesis, FIAN (1959); Sayasov Yu. S., and Melnikov V. K., Review OIYaI, P-201 (1958).
13. Bulgakov B. V., Oscillations, Gostekhizdat, 1954.
14. Orlov Yu. F., ZhETF **32,** 130 (1957).
15. Blachman N., Rev. Sci. Instr. **21,** 908 (1950). [Russian translation in PSF, No. 2, 52 (1952)].
16. Kharbevich A. A., Spectra and their analysis, Gostekhizdat, 1957.
17. Goldin L. L., and Koshkarev D. G., PTE, No. 3, 3 (1957).
18. Rabinovich M. S., Bases of the theory of the synchrophasotrons with straight sections, Thesis, FIAN (1952); FIAN Report X, 23 (1958).
19. Danilkin I. S., and Rabinovich M. S., ZhTF **27,** 1558 (1957).
20. Courant E. D., Livingston M. S., and Snyder H. S., Phys. Rev. **88,** 1190 (1952). [Russian translation in PSF, No. 11, 169 (1954)].
21. Kolomensky A. A., and Sabsovich L. L., ZhTF **26,** 576 (1956); CERN Symposium **1,** 112 (1956).

22. Johnsen K., in "Lectures on the Theory and Design of an Alternating Gradient Synchrotron", Geneva (1953).
23. Symon K., and Sessler A., CERN Symposium **1,** 44 (1956).
24. Johnsen K., and Schmelzer C., CERN Symposium **1,** 395 (1956).
25. Regenstreif E., CERN Report, 59-29 (1959); Schnell W., Proceedings of the International Conference on Accelerators, CERN, Geneva, 485 (1959).
26. Burstein E. L., and Belov N. E., Proceedings of the RAIAN **1,** 3, 3 (1960); Mints A. L., Vasiliev A. A., Vodopyanov F. A., Kuzmin A. A., Kuzmin V. F., Rubchinskij E. M., Proceedings of the International Conference on Accelerators, CERN, Geneva, 427 (1959).
27. Kolomensky A. A., Petukhov V. A., Rabinovich M. S., FIAN Report, RF-54 (1953); Manual "Some questions on the theory of cyclic accelerators" Edition AN SSSR (1955); Kolomensky A. A., ZhTF **26,** 740 (1956).
28. Vladimirsky V. V., and Tarasov E. K., Manual "Some questions on the theory of cyclic accelerators", Edition AN SSSR (1955).
29. Vladimirsky V. V., Komar E. G., and Mints A. L., CERN Symposium **1,** 118 (1956).
30. Kerst D., and Serber R., Phys. Rev. **60,** 53 (1941).
31. Terletsky Ya. P., ZhETF **11,** 96 (1941); Journ. Phys. **9,** 159 (1945).
32. Kerst D., UFN **XXIV,** 181 (1944).
33. Grinberg A. P., UFN **XXVII,** 31 (1945).
34. Ivanenko D. D., and Pomeranchuk I. Ya., DAN SSSR **44,** 343 (1944).

CHAPTER V

1. Schott G., Electromagnetic Radiation, Cambridge, 1912.
2. Pomeranchuk I. Ya., ZhETF **9,** 915 (1939).
3. Ivanenko D. D., and Pomeranchuk I. Ya., DAN SSSR **44,** 343 (1944).
4. Artsimovich L. A., and Pomeranchuk I. Ya., ZhETF **16,** 379 (1946).
5. Schwinger J., Phys. Rev. **70,** 798 (1946).
6. Landau L. D., and Lifshits E. M., Field Theory, Fizmatgiz, 1960.
7. Vladimirsky V. V., ZhETF **18,** 392 (1948).
8. Ivanenko D. D., and Sokolov A. A., Classical Field Theory, Gostekhizdat, 1949.
9. Bohm D., and Foldy L., Phys. Rev. **70,** 249 (1946); Journ. Phys. **10,** 523 (1946).
10. Sokolov A. A., and Ternov I. M., DAN SSSR **97,** 823 (1954); ZhETF **28,** 431 (1955).
11. Sands M., Phys. Rev. **97,** 470 (1955).
12. Kolomensky A. A., and Lebedev A. N., DAN SSSR **106,** 87 (1956); ZhETF **30,** 207, 1161 (1956).
13. Kolomensky A. A. and Lebedev A. N., CERN Symposium **1,** 447, Geneva (1956); Nuovo Cimento Suppl. **7** (1), **43** (1958).
14. Kolomensky A. A., Investigations on the theory of particle motion in modern cyclic accelerators, Thesis, FIAN (1956); Proceedings FIAN **XIII,** 3, Edition AN SSSR (1960).
15. MacMillan E., Phys. Rev. **68,** 144 (1945).
16. Schiff L. J., Rev. Sci. Instr. **17,** 6 (1946).

17. Nodvick J., and Saxon D., Phys. Rev. **96,** 180 (1954).
18. Johansen L. V., and Rabinovich M. S., ZhETF **35,** 1013 (1958); **37,** 118 (1959).
19. Prokhorov A. M., Radiotechnique and Electronics **1,** 71 (1956).
20. Bogolyubov N. N., and Mitropolsky Yu. A., Asymptotic methods in the theory of non-linear oscillations, Fizmatgiz (1958).
21. Kolomensky A. A., and Lebedev A. N., AE **8,** 553 (1960).
22. Smirnov V. I., Course on higher mathematics, Gostekhizdat. 1953. Volume III. part 2, page 438.
23. Stepanov V. V., Course on differential equations, Gostekhizdat. 1953, page 192.
24. Kolomensky A. A., and Lebedev A. N., AE **5,** 554 (1958).
25. Robinson K. W., Report CEA (Cambridge Electron Accelerator) 1956, not published; Phys. Rev. **111,** 373 (1958).
26. Orlov Yu. F., and Tarasov E. K., ZhETF **34,** 651 (1958); PTE. No. 5. 15 (1958).
27. Orlov Yu. F., Tarasov E. K., Khejfets S. A., Proceedings of the International Conference on Accelerators, CERN, Geneva, 306 (1959).
28. Akhiezer A. I., and Berestetsky V. B., Quantum Electrodynamics. Gostekhizdat, 1953, paragraph 32.
29. Bloch F., and Nordsieck A., Phys. Rev. **52,** 54 (1937); Glauber K., Phys. Rev. **84,** 395 (1951).
30. Landau L. D., and Lifshits E. M., Statistical Physics, Gostekhizdat, 1951. page 373.
31. Watson G. N., A treatise on the theory of Bessel functions, University Press Cambridge 1944 and IL 1949 page 424.
32. Ryzhik I. M., and Gradstein I. O., Tables of integrals, sums, series and products, Gostekhizdat, 1951, page 332, and Berlin, VEB Deutscher Verlag der Wissenschaften, 1963.
33. Livingston M. S., CERN Symposium **1,** 335 (1956); Jentschke W., ibid., 339.
34. Moroz E. M., Rabinovich M. S., AE **4,** 128 (1958).
35. Elder F. R., Langmuir R. V., Pollock H. C., Phys. Rev. **74,** 52 (1948); Ado Yu. M., and Cherenkov P. A., DAN SSSR **110,** 35 (1956).
36. Korolev F. A., Ershov A. G., Kulikov O. F., DAN SSSR **134,** 314 (1960).

CHAPTER VI

1. Blachman N., and Courant E., Phys. Rev. **74,** 140 (1948) [Russian translation in PSF, No. 2, 75 (1952)]; Phys. Rev. **75,** 315 (1949); Rev. Sci. Instr. **20,** 596 (1949); **24,** 836 (1953).
2. Kolmogorov A. N., UMN **5,** 30 (1938).
3. Leontovich M. A., Statistical Physics, Gostekhizdat, 1944.
4. Chandrasekar S., Stochastic problems in physics and astronomy, IL, 1947.
5. Jahnke E., Emde F., Tables of functions, Dover, New-York, 1945 and Fizmatgiz, 1959.
6. Berestetsky V. B., Goldin L. L., Koshkarev D. G., PTE, No. 3, 26 (1956).
7. Rossi B., High energy particles, Prentice Hall, New York, 1952 and Gostekhizdat, 1955.

8. Greenberg J. M., and Berlin T. H., Rev. Sci. Instr. **22,** 293 (1951) [Translation in PSF, No. 2, 81 (1952)].
9. Moravcsik M. J., and Sellen J. M., Rev. Sci. Instr. **26,** 1158 (1955); Moravcsik M. J., Phys. Rev. **100,** 1009 (1955).
10. Kramers H. A., Physika **7,** 285 (1940).
11. Robinson K., Report CEA-69 (1958); Proceedings of the International Conference on Accelerators, CERN, Geneva, 293 (1959).
12. Christy R., Synchrotron Beam Loss due to Quantum Fluctuations in the Radiation, California Institute of Technology Report (1957).
13. Matveev A. N., ZhETF **33,** 1254 (1957).
14. Sands M., Nuovo Cimento **XV** (4), 599 (1960).
15. Goldin L. L., and Koshkarev D. G., PTE, No. 3, 3 (1937).
16. Orlov Yu. F., Khejfets S. A., Proceedings of the International Conference on Accelerators, CERN, Geneva, page 303 (1959).
17. Lebedev N. N., Special Functions and their Application, Gostekhizdat, 1953.
18. Goldin L. L. and Ioffe R. A., Proceedings of the International Conference on Accelerators, CERN, Geneva, 653 (1959).
19. Bernardini C., CERN Symposium **1,** 463 (1956).
20. Bolotovsky B. M., and Fateev A. P., ZhETF **33,** 304 (1957).
21. Molière G., Z. Naturforsch. **2A,** 133 (1947).
22. Kotov V. I., Kuznetsov A. B., Rubin N. B., Manual "Accelerators", Atomizdat (1961).
23. Landa P. S., ZhETF **40,** (4), 1119 (1961).

CHAPTER VII

1. Livingston M., Rose M., Nahmias M., Collection of Reports "The Cyclotron", Gostekhizdat (1948), see also Livingston M. S., Blewett J. P., Particle accelerators, Mac Graw Hill New-York, 1962.
2. Grinberg A. P., Methods of accelerating charged particles, Gostekhizdat, 1950.
3. Rose M. E., Phys. Rev. **53,** 392 (Russian translation, see 1).
4. Khurgin Ya. L., DAN SSSR **19,** 237 (1938).
5. Thomas L. H., Phys. Rev. **54,** 580, 588 (1938).
6. King N. M., and Walkinshaw W., Nucl. Instr. **2,** 287 (1958).
7. Vasilevskaya D. P., Glazov A. A., Danilov V. N., Denisov Yu. N., Dzhelepov V. P., Dmitrievsky V. P., Zamolodchikov B. I., Zaplatin N. L., Kolga V. V., Kropin A. A., Lyu Ne-guan, Pybalko V. C., Savenkov A. L., Sarkisian L. A., AE **6,** 657 (1959); **8,** 189 (1960).
8. Dunn P. D., Mallet L. B., Pickavance T. G., Walkinshaw W., Wilkins J. J., CERN Symposium **1,** 9 (1956).
9. Moroz E. M., Rabinovich M. S., PTE, No. 1, 15 (1957).
10. Efremov D. V., Mesheryakov M. G., Mints A. L., Dzhelepov B. P., Ivanov P. P., Katyshev V. S., Komar E. G., Monoszon N. A., Nevyazhsky I. H., Polyakov B. I., Chestnoj A., CERN Symposium **1,** 148 (1956); AE **4,** 5 (1956).

11. Livingston M. S., Rev. Mod. Phys. **18,** 3 (1946).
12. Bohm D., and Foldy L., Phys. Rev. **72,** 649 (1947).
13. Rabinovich M. S., Theory of modern resonance accelerators. Thesis, FIAN (1948).
14. Richardson J. R., Wright B. T., Lofgren E. J., Peters B., Phys. Rev. **73,** 424 (1948).
15. Henrich L. R., Sewell D. C., Vale J., Rev. Sci. Instr. **20,** 887 (1949).
16. Tuck J. L., Teng L. C., Phys. Rev. **81,** 305 (1951).
17. Dmitrievsky V. P., Danilov V. I., Denisov Yu. N., PTE, No. 1, 11 (1957).
18. Veksler B. I., DAN SSSR **43,** 346 (1944); Journ. Phys. **9,** 153 (1945).
19. Kolomensky A. A., Theory of the harmonic number resonance accelerator (microtron), Thesis, FIAN (1950); ZhTF **30,** 1347 (1960).
20. Gelfond A. O., Calculation of finite differences, Gostekhizdat, 1952.
21. Kapitza S. P., Bykov V. P., Melekhin V. N., ZhETF **39,** 997 (1960).
22. Moroz E. M., DAN SSSR **108,** 436 (1956).
23. Roberts A., Report of the Rochester University, No. 40-8055 (1958).
24. Courant E. D., Livingston M. S., Snyder H. S., Phys. Rev. **88,** 1190 (1952). [Russian translation in PSF, No. 11, 169 (1954)].
25. Kolomensky A. A., Proceedings FIAN **XIII,** 3 (1960).
26. Regenstreif E., The CERN Proton Synchrotron, CERN Report 59-29. Geneva (1959).
27. Crane H. R., Phys. Rev. **69,** 542 (1946).
28. Rabinovich M. S., Thesis, FIAN (1952); Proc. FIAN **X,** 23 (1958).
29. Blachman N., Courant E., Rev. Sci. Instr. **20,** 596 (1949); Blachman N. M., Rev. Sci. Instr. **22,** 569 (1951).
30. Crosbie E., et al., Proceedings of the International Conference on Accelerators, CERN Geneva, 42 (1959).
31. Kolomensky A. A., Petukhov V. A., Rabinovich M. S., Manual "Some questions of the theory of cyclic accelerators", Edition AN SSSR, 1955, page 7. PTE, No. 2, 26 (1956).
32. Symon K. R., Kerst D. W., Jones L. W., Laslett L. J., Terwilliger K. M., Phys. Rev. **103,** 1837 (1956). [Russian translation in UFN **LXI,** 613 (1957)].
33. Kolomensky A. A., ZhETF **33,** 298 (1957).
34. Ohkawa T., Rev. Sci. Instr. **29,** 108 (1958).
35. Kannunikov V. N., Kolomensky A. A., Lebedev A. N., Obchinnikov E. V., Stolov A. M., Titov V. A., Fateev A. P., Yablokov B. N., Proceedings of the International Conference on Accelerators, CERN, Geneva, 89 (1959).
36. Haxby R. O., Laslett L. J., Mils F. E., Peterson F. L., Rowe E. M., Wallenmeyer W. A., Proceedings of the International Conference on Accelerators, CERN, Geneva, 75 (1959).
37. Kolomensky A. A., AE **3,** 492 (1957).
38. Sansone G., Ordinary differential equations, IL, 1954 and Equazioni differenziali nel campo reale, Bologna 1948.
39. Kolomensky A. A., and Lebedev A. N., AE **8,** 553 (1960).
40. Cole F. T., Proceedings of the International Conference on Accelerators, CERN, Geneva, 89 (1959).
41. Fateev A. P., ZhTF **31,** 238 (1961).
42. Symon K., and Sessler A., CERN Symposium **1,** 44 (1956).

43. Vogt-Nilsen N., CERN Report 58-9 (1958).
44. Lebedev A. N., AE **9,** 189 (1960); Questions of the theory of acceleration and storage systems for high intensity beams, Thesis, FIAN (1960).
45. O'Neil G. K., Proceedings of the International Conference on Accelerators, CERN, Geneva, 125 (1959).
46. Burstein E. L., Veksler V. I., Kolomensky A. A., Manual "Some questions of the theory of cyclic accelerators", Edition AN SSSR, 1955, page 3.
47. Kolomensky A. A., Lebedev A. N., Proceedings of the International Conference on Accelerators, CERN, Geneva, 184 (1959).
48. Keller R., ibid page 187.
49. Barbier M., ibid page 636.
50. Cole F. T., Haxby R. O., et al., Rev. Sci. Instr. **28,** 403 (1957); Petukhov V. A., Gabanetz I,, Zhuravlev A. A., Karmasin M., Kotov V. I., Myae E. A., Obukhov Yu. L., Sokhor V., Tsirak Yu., Benda F., Dobiash I., Marek M., Svetov L. V., Fukatko T., ZhTF **31,** 1253 (1961).

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